On the gauge orbit space stratification: a review

2002 J. Phys. A: Math. Gen. 35 R1

(http://iopscience.iop.org/0305-4470/35/28/201)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) R1–R50 PII: S0305-4470(02)33597-2

TOPICAL REVIEW

On the gauge orbit space stratification: a review

G Rudolph[1], **M Schmidt**[1] **and I P Volobuev**[2]

[1] Institute for Theoretical Physics, University of Leipzig, Augustusplatz 10, 04109 Leipzig, Germany

[2] Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119992 Moscow, Russia

Received 4 March 2002
Published 5 July 2002
Online at stacks.iop.org/JPhysA/35/R1

Abstract

First, we review the basic mathematical structures and results concerning the gauge orbit space stratification. This includes general properties of the gauge group action, fibre bundle structures induced by this action, basic properties of the stratification and the natural Riemannian structures of the strata. In the second part, we study the stratification for theories with gauge group SU(n) in spacetime dimension 4. We develop a general method for determining the orbit types and their partial ordering, based on the 1–1 correspondence between orbit types and holonomy-induced Howe subbundles of the underlying principal SU(n)-bundle. We show that the orbit types are classified by certain cohomology elements of spacetime satisfying two relations and that the partial ordering is characterized by a system of algebraic equations. Moreover, operations for generating direct successors and direct predecessors are formulated, which allow one to construct the set of orbit types, starting from the principal type. Finally, we discuss an application to nodal configurations in Yang–Mills–Chern–Simons theory.

PACS numbers: 11.15.−q, 02.20.−a, 02.40.−k, 11.10.−z

1. Introduction

One of the basic principles of modern theoretical physics is the principle of local gauge invariance. Its application to the theory of particle interactions gave rise to the standard model, which proved to be a success from both theoretical and phenomenological points of view. The most impressive results of the model were obtained within the perturbation theory, which works well for high energy processes. On the other hand, the low energy hadron physics, in particular, the quark confinement, turns out to be dominated by nonperturbative effects, for which there is no rigorous theoretical explanation yet. To study them, a variety of different concepts and mathematical methods has been developed. In particular, for some aspects methods of differential geometry and algebraic topology seem to be unavoidable. This is certainly true if one wants to investigate the structure of the configuration space of a gauge theory—the space of gauge group orbits. In general, this space possesses not only orbits of the so-called principal type, but also orbits of other types, which may give rise to singularities. This stratified structure of the gauge orbit space is believed to be of importance for both classical and quantum properties of non-Abelian gauge theories in the nonperturbative approach. Let us discuss some aspects indicating its physical relevance.

First, studying the geometry and topology of the generic (principal) stratum, one gets an intrinsic topological interpretation of the Gribov-ambiguity [40, 70]. We stress that the problem of finding all Gribov copies has been discussed within specific models, see, e.g., [57]. For a detailed analysis in the case of two-dimensional cylindrical spacetime (including the Hamiltonian path integral) we refer to [69]. Investigating the topology of the determinant line bundle over the generic stratum, one gets an understanding of anomalies in terms of the family index theorem [3, 8], see also [22] for the Hamiltonian approach. In particular, one gets anomalies of purely topological type [78], which cannot be seen by perturbative quantum field theory. Moreover, there are partial results and conjectures concerning the relevance of nongeneric strata. First, generally speaking, nongeneric gauge orbits affect the classical motion on the orbit space due to boundary conditions and, in this way, may produce nontrivial contributions to the path integral. They may lead to localization of certain quantum states, as was suggested by finite-dimensional examples [29]. Further, the gauge field configurations belonging to nongeneric orbits can possess a magnetic charge, i.e. they can be considered as a kind of magnetic monopole configuration. Following t'Hooft [74], these could be responsible for quark confinement. The role of these configurations was investigated within the framework of Schrödinger quantum mechanics on the gauge orbit space of topological Chern–Simons theory in [4], see also [5] for an approach to four-dimensional Yang–Mills theories with θ-term. Within t'Hooft's concept, the idea of Abelian projection is of special importance and has been discussed by many authors. Recently, this concept was studied within the setting of quantum field theory at finite temperature on the 4-torus [35, 36]. There, a hierarchy of defects, which should be related to the gauge orbit space structure, was discovered. Finally, let us also mention that the existence of additional anomalies corresponding to non-generic strata was suggested, see [44].

Most of the problems mentioned here are still awaiting a systematic investigation. For that purpose, a deeper insight into the structure of the gauge orbit space is necessary. In a series of papers [65–67] we have made a new step in this direction. We have given a complete solution to the problem of determining the strata that are present in the gauge orbit space for $\mathrm{SU}(n)$ gauge theories in compact Euclidean spacetime of dimension $d = 2, 3, 4$. Our analysis is based on the results of Kondracki and Rogulski [54], where the general structure of the full gauge orbit space was investigated for the first time in detail. In particular, it was shown that the gauge orbit space is a stratified topological space. Moreover, these authors found the basic relation between orbit types and certain bundle reductions, which we are using. We note that this relation was also observed in [43].

We mention that there is an approach based upon parametrizing the full gauge orbit space by a so-called fundamental domain, characterized by the fact that, up to identifications on the boundary, it is intersected by every gauge orbit exactly once, see [26, 38, 76, 77, 79] and references therein. However, for the study of the stratified structure of the gauge orbit space, this concept seems not to be efficient.

Finally, we note that the stratification structure for gauge theories within the Ashtekar approach has also been clarified, see [33].

This review is organized as follows. In the first part, the basic mathematical structures and results concerning the gauge orbit space stratification are discussed. In section 2, we briefly recall the setup and sketch the basic properties of the gauge group action, including a slice theorem and an approximation theorem. In section 3, the fibre bundle structures induced by this action are investigated. Next, in section 4, basic properties of the stratification are derived and, in section 5, the natural Riemannian structures of the strata are discussed. This concludes the general part of the review. In the remaining part, we specify the gauge group to be SU(n) and spacetime to be of dimension less than or equal to 4. Under these assumptions, the strata can be classified by characteristic classes of certain reductions of the principal bundle the theory is defined on. This will be explained in section 6. In section 7, we show how the natural partial ordering of strata, which contains information on how the strata are linked, can be read off from algebraic relations between the characteristic classes. Finally, we discuss the case of gauge group SU(2) for some 4-manifolds in detail and present an application to nodal configurations in topological Chern–Simons theory. For the convenience of the reader, we have added two appendices on aspects of bundle theory and algebraic topology used in the text, as well as an appendix in which we explain how to construct the Postnikov towers of the classifying spaces relevant for the classification of orbit types.

2. Basics

2.1. Setup

In what follows, we assume that the reader is familiar with the standard formulation of gauge theories in terms of fibre bundles and connections [25, 28, 75]. Thus, let M be a compact connected orientable Riemannian manifold, let G be a compact connected linear Lie group with Lie algebra $\mathfrak{g}$ and let P be a smooth locally trivial principal G-bundle over M. In physical terms, M is a model of spacetime and G is the gauge group.

For any vector bundle E, let $W^k(E)$ denote the Hilbert space of cross sections of E of Sobolev class k. For generalities on such spaces, see [61]; for the application of these techniques to gauge theories, see [58]. Let $\mathcal{C}$ denote the subspace of $W^k(\mathrm{T}^*P \otimes \mathfrak{g})$ of connection forms on P of Sobolev class k and let $\mathcal{G}$ denote the closure of the group of smooth G-space morphisms $P \to G$ in $W^{k+1}(P, \mathrm{gl}(n, \mathbb{C}))$. Here n is chosen so that $G \subseteq \mathrm{gl}(n, \mathbb{C})$. In physics, elements A of $\mathcal{C}$ represent gauge potentials, whereas elements g of $\mathcal{G}$ represent local gauge transformations, acting by

$$A^{(g)} = \mathrm{Ad}(g^{-1})A + g^{-1}\,\mathrm{d}g. \tag{1}$$

The space $\mathcal{C}$ is an affine separable Hilbert space with translational vector space

$$\mathcal{T} = W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P)$$

where $\mathrm{Ad}P$ denotes the associated bundle $P \times_G \mathfrak{g}$. Throughout the review, we will assume $k > \dim(M)/2 + 1$. Then the Sobolev lemma ensures that multiplication of a W^{k+1}-function by a W^l-function, $\dim(M)/2 < l \leqslant k$, yields a W^l-function. It follows that $\mathcal{G}$ is a group, acting via (1) on $\mathcal{C}$. In fact, one can prove that $\mathcal{G}$ is a Hilbert–Lie group with Lie algebra

$$\mathrm{L}\mathcal{G} = W^{k+1}(\mathrm{Ad}P)$$

and exponential mapping

$$\exp_{\mathcal{G}}(\xi)(p) = \exp_G(\xi(p)) \qquad \forall \xi \in \mathrm{L}\mathcal{G} \quad p \in P \tag{2}$$

and that the action is smooth [59, 60, 70].

It should be noted that for both $\mathcal{T}$ and $\mathrm{L}\mathcal{G}$, identification of sections in associated bundles with the corresponding G-equivariant horizontal forms on P is understood. We will stick to this identification throughout the review. Also note that the elements of $\mathcal{C}$ and $\mathcal{G}$ are C^1 and C^2, respectively. In particular, $\mathcal{G}$ may be viewed as consisting of vertical automorphisms of P of class C^2 or of sections of class C^2 in the associated fibre bundle $P \times_G G$ [32].

The gauge orbit space is

$$\mathcal{M} := \mathcal{C}/\mathcal{G}$$

which is, at this stage, just a topological quotient. It will be equipped with additional structure later. Note that $\mathcal{M}$ is the space of classes of gauge equivalent potentials—the 'true' configuration space.

The scalar products on the Hilbert spaces $\mathrm{L}\mathcal{G}$ and $\mathcal{T}$, respectively, are not intrinsic. Their only purpose is to define the topology. The geometry of these spaces is defined by L^2-scalar products, induced from the Riemannian metric on M and an $\mathrm{Ad}(G)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ as follows:

$$(\xi,\eta)_0 := \int_M \langle \xi, *\eta\rangle \quad \xi,\eta\in \mathrm{L}\mathcal{G} \qquad (X,Y)_0 := \int_M \langle X\wedge *Y\rangle \quad X,Y\in\mathcal{T}$$

respectively. Here $*$ denotes the Hodge duality operator. Both of these scalar products are invariant under the adjoint action of $\mathcal{G}$.

Since $\mathcal{C}$ is affine with translational vector space $\mathcal{T}$, we have

$$\mathrm{T}\mathcal{C} = \mathcal{C}\times\mathcal{T}. \tag{3}$$

In particular, any smooth assignment of a scalar product in $\mathcal{T}$ to the elements of $\mathcal{C}$ defines a Riemannian metric on $\mathcal{C}$. Examples are:

(i) The constant assignment $A\mapsto(\cdot,\cdot)_0$ defines the natural (weak) L^2-metric γ^0. It is invariant under the induced action of $\mathcal{G}$ on $\mathcal{T}$, given by

$$X^{(g)} = \mathrm{Ad}(g^{-1})X.$$

(ii) The assignment $A\mapsto\gamma^k_A$, induced from

$$\gamma^k_A(X,Y) := \sum_{l=0}^{k}\big([\widetilde{\nabla}_A]^l X, [\widetilde{\nabla}_A]^l Y\big)_0 \qquad X,Y\in C^\infty(\mathrm{T}^*M\otimes \mathrm{Ad}P) \tag{4}$$

by prolongation to $\mathcal{T}$, defines a natural metric γ^k. Here

$$\widetilde{\nabla}_A : C^\infty(\mathrm{T}^*M^{\otimes l}\otimes\mathrm{Ad}P)\to C^\infty\big(\mathrm{T}^*M^{\otimes(l+1)}\otimes\mathrm{Ad}P\big) \qquad \alpha\mapsto\nabla^{LC}\alpha+[A,\alpha]$$

where ∇^{LC} is the Levi-Civita connection of the Riemannian metric on M and

$$[A,\alpha](X_0,X_1,\ldots,X_l) = [A(X_0),\alpha(X_1,\ldots,X_l)].$$

The norm on $\mathcal{T}$ defined by the scalar products γ^k_A, $A\in\mathcal{C}$, is equivalent to the W^k-norm [27]. Therefore, γ^k is a strong metric. Moreover, due to

$$\big(\widetilde{\nabla}_{A^{(g)}}\big)^l = \mathrm{Ad}(g^{-1})(\widetilde{\nabla}_A)^l\,\mathrm{Ad}(g)$$

it is $\mathcal{G}$-invariant, $\gamma^k_{A^{(g)}}(X^{(g)},Y^{(g)}) = \gamma^k_A(X,Y)$.

(iii) Let us remark that one can construct further $\mathcal{G}$-invariant metrics using the Laplacian $\Box_A = \nabla_A^*\nabla_A+\nabla_A\nabla_A^*$ as

$$\eta^k_A(X,Y) = \big((1+\Box_A)^{k/2}X,(1+\Box_A)^{k/2}Y\big)_0 \tag{5}$$

where $(1+\Box_A)^{k/2}$ is defined via functional calculus. For some specific examples, like the principal SU(2)-bundle of second Chern class ('instanton number') $c_2=1$ over $\mathbb{C}\mathrm{P}^2$, the restriction of η^2 to the moduli space of irreducible self-dual connections was studied in detail, see [42] and references therein. We do not comment on this here.

Next, for $A \in \mathcal{C}$, consider the operator of covariant derivative w.r.t. A,

$$\nabla_A : W^{k+1}(\mathrm{Ad}P) \to W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P).$$

Its formal adjoint w.r.t. the L^2-scalar product is the bounded linear operator

$$\nabla_A^* : W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P) \to W^{k-1}(\mathrm{Ad}P)$$

defined by

$$(\nabla_A \xi, X)_0 = (\xi, \nabla_A^* X)_0 \qquad \forall \xi \in C^\infty(\mathrm{Ad}P) \quad X \in C^\infty(\mathrm{T}^*M \otimes \mathrm{Ad}P).$$

Composition then yields a bounded linear operator

$$\Delta_A = \nabla_A^* \nabla_A : W^{k+1}(\mathrm{Ad}P) \to W^{k-1}(\mathrm{Ad}P).$$

In the following, instead of $W^l(\mathrm{Ad}P)$ or $W^l(\mathrm{T}^*M \otimes \mathrm{Ad}P)$ we shall often write W^l, because the bundle in which the sections are taken can be read off unambiguously from the operators under consideration. Moreover, the pure symbols ∇_A, ∇_A^*, Δ_A always stand for the maps $\nabla_A|W^{k+1}$, $\nabla_A^*|W^k$ and $\Delta_A|W^{k+1}$ with k fixed, whereas, for example, $\nabla_A|W^{l+1}$ means that ∇_A is viewed as an operator $W^{l+1} \to W^l$ (where $\dim(M)/2 < l \leqslant k$).

Note that the maps

$$\mathcal{C} \to \mathrm{B}(W^{k+1}, W^k) \quad A \mapsto \nabla_A \qquad \mathcal{C} \to \mathrm{B}(W^k, W^{k-1}) \quad A \mapsto \nabla_A^*$$

are continuous linear. Hence, the map

$$\mathcal{C} \to \mathrm{B}(W^{k+1}, W^{k-1}) \qquad A \mapsto \Delta_A$$

is continuous. Since it factorizes into continuous linear maps and composition of operators, it is even smooth. Moreover, we note the following equivariance properties:

$$D_{A^{(g)}} = \mathrm{Ad}(g^{-1}) D_A \,\mathrm{Ad}(g) \qquad \forall A \in \mathcal{C} \quad g \in \mathcal{G} \tag{6}$$

where D stands for ∇, ∇^* and Δ, respectively.

2.2. *Stabilizers*

Recall that the stabilizer (or isotropy subgroup) of $A \in \mathcal{C}$ w.r.t. the action of $\mathcal{G}$ is the subgroup

$$\mathcal{G}_A := \left\{ g \in \mathcal{G} : A^{(g)} = A \right\}$$

of $\mathcal{G}$. It is determined by the holonomy of A. Indeed, $g \in \mathcal{G}_A$ iff g is constant on any curve horizontal with respect to A. Thus

$$\mathcal{G}_A = \left\{ g \in \mathcal{G} : g\big|_{P_{A,p_0}} = \mathrm{const} \right\} \tag{7}$$

where P_{A,p_0} denotes the holonomy bundle of A based at $p_0 \in P$. Note that P_{A,p_0} is of class C^2, because A is C^1.

Let $\xi \in \mathrm{L}\mathcal{G}$. We have

$$\nabla_A \xi = 0 \quad \Leftrightarrow \quad \xi\big|_{P_{A,p_0}} = \mathrm{const} \quad \Leftrightarrow \quad \exp_{\mathcal{G}}(\xi)\big|_{P_{A,p_0}} = \mathrm{const}$$

where the second equivalence is due to (2). Thus

$$\exp_{\mathcal{G}}(\mathrm{L}\mathcal{G}) \cap \mathcal{G}_A = \exp_{\mathcal{G}}(\ker(\nabla_A)).$$

Since $\ker(\nabla_A)$ is a closed subspace of the Hilbert space $\mathrm{L}\mathcal{G}$, the rhs is a submanifold of $\mathcal{G}$. Since the lhs is a neighbourhood of e in $\mathcal{G}_A$, it follows that $\mathcal{G}_A$ is a Lie subgroup of $\mathcal{G}$ with Lie algebra

$$\mathrm{L}\mathcal{G}_A = \ker(\nabla_A) = \left\{ \xi \in \mathrm{L}\mathcal{G} : \xi\big|_{P_{A,p_0}} = \mathrm{const} \right\} \tag{8}$$

see [14, section III.1.3]. Next, consider the natural group homomorphism

$$\Phi_{p_0} : \mathcal{G} \to G, g \mapsto g(p_0)$$

(the value of g at a point is of course well defined). Since convergence in W^{k+1}, by our choice of k, implies pointwise convergence, Φ_{p_0} is continuous, hence smooth. Due to (7), the restriction of Φ_{p_0} to the subgroup $\mathcal{G}_A$ is injective, hence a Lie group isomorphism onto its image. The image is

$$\Phi_{p_0}(\mathcal{G}_A) = \mathrm{C}_G\left(H_{A,p_0}\right)$$

where H_{A,p_0} denotes the holonomy group of A based at p_0. To see this, recall that H_{A,p_0} is the structure group of P_{A,p_0}. Thus, inclusion from left to right is due to equivariance of the elements of $\mathcal{G}$. For the converse inclusion it suffices to note that for any $a \in \mathrm{C}_G(H_{A,p_0})$, the function on P_{A,p_0} with constant value a is equivariant and, hence, can be equivariantly prolonged to P, thus becoming an element of $\mathcal{G}_A$.

Let us summarize.

Theorem 2.1 (Stabilizer theorem)**.** *$\mathcal{G}_A$ is a compact Lie subgroup of $\mathcal{G}$ with Lie algebra given by (8). Through Φ_{p_0}, $\mathcal{G}_A$ is isomorphic to $\mathrm{C}_G\big(H_{A,p_0}\big)$.*

As an immediate consequence of the fact that $\mathcal{G}_A$ is an (embedded) Lie subgroup, the projection $\mathcal{G} \to \mathcal{G}/\mathcal{G}_A$ defines a locally trivial principal bundle [14, section 6.2.4].

In [60] it was shown that the map $\mathcal{C} \times \mathcal{G} \to \mathcal{C} \times \mathcal{C}$, $(A, g) \mapsto (A, A^{(g)})$, is closed. It follows [16, III, section 4]

Theorem 2.2. *The action of $\mathcal{G}$ on $\mathcal{C}$ is proper.*

The immediate consequences are:

(i) The orbits of the action of $\mathcal{G}$ on $\mathcal{C}$ are closed.
(ii) The orbit space $\mathcal{M}$ is Hausdorff.

A different proof of theorem 2.2 was given in [54]. By assigning to $A \in \mathcal{C}$ a W^k-Riemannian metric on P,

$$h_A(u, v) = h_M(\pi_* u, \pi_* v) + \langle A(u), A(v)\rangle \qquad u, v \in \mathrm{T}_p P \quad p \in P$$

where h_M is the Riemannian metric on M, a homeomorphism of $\mathcal{C}$ onto a closed submanifold of the manifold $\mathrm{Met}^k(P)$ of W^k-Riemannian metrics on P is constructed (it is even a diffeomorphism into). $\mathrm{Met}^k(P)$ is acted upon by the topological group $\mathrm{Diff}^{k+1}(P)$ of W^{k+1}-diffeomorphisms of P. $\mathrm{Diff}^{k+1}(P)$ is known to be a smooth manifold, but not a Lie group. The action is known to be smooth and proper [17, 27, 31]. It is shown in [54] that $\mathcal{G}$ is a closed topological subgroup of $\mathrm{Diff}^{k+1}(P)$ (it is even a submanifold) and that the embedding $\mathcal{C} \to \mathrm{Met}^k(P)$ is equivariant. Thus, properness carries over from the action of $\mathrm{Diff}^{k+1}(P)$ on $\mathrm{Met}^k(P)$ to that of $\mathcal{G}$ on $\mathcal{C}$.

Note that compactness of stabilizers is not needed in the second proof. Rather, it is a consequence of properness of the action.

2.3. Orbit types

According to $\mathcal{G}_{A^{(g)}} = g^{-1}\mathcal{G}_A g$, the stabilizers along an orbit $x \in \mathcal{M}$ form a conjugacy class in $\mathcal{G}$. This class is called the orbit type of x and is denoted by $\mathrm{Type}(x)$. The set of orbit types carries a natural partial ordering: $\sigma \leqslant \sigma'$ iff there exist representatives S of σ and S' of σ' such that $S \supseteq S'$. Then for any pair of representatives S, S' there exists $g \in \mathcal{G}$ such that

$S \supseteq aS'a^{-1}$. One says that S' is subconjugate to S. Note that, although this definition of the partial ordering of orbit types is the usual one [14, 19], it is not consistent with [54], where the inverse partial ordering is used.

We are going to characterize orbit types in terms of certain bundle reductions of P, see also [43] for a similar approach. For that purpose, let us consider, for a moment, smooth connections and smooth local gauge transformations. Recall that a subgroup of G that can be written as a centralizer is usually called a *Howe subgroup*. This is due to the fact that such a subgroup, together with its centralizer, forms a reductive dual pair, a notion introduced by Howe [45–47]. According to that, let us call a bundle reduction of P to a Howe subgroup of G a *Howe subbundle*. (All bundle reductions are assumed to be smooth.) As any subgroup $H \subseteq G$ generates a Howe subgroup $\widetilde{H}$ (containing H) by $\widetilde{H} = \mathrm{C}_G^2(H)$, any bundle reduction Q of P to H generates a Howe subbundle $\widetilde{Q}$ (containing Q) by extending the structure group to $\widetilde{H}$,

$$\widetilde{Q} = Q\widetilde{H}.$$

In particular, a connection A generates a Howe subbundle $\widetilde{P}_{A,p_0}$ through its holonomy bundle. In [54], $\widetilde{P}_{A,p_0}$ was called the evolution bundle of A. Since an element of G that commutes with H_{A,p_0} still commutes with $\widetilde{H}_{A,p_0}$, a gauge transformation that is constant on P_{A,p_0} is still constant on $\widetilde{P}_{A,p_0}$. Thus

$$\mathcal{G}_A = \left\{ g \in \mathcal{G} : g \Big|_{\widetilde{P}_{A,p_0}} = \mathrm{const} \right\}. \tag{9}$$

We claim that $\widetilde{P}_{A,p_0}$ consists of all $p \in P$ obeying

$$g(p) = g(p_0) \qquad \forall g \in \mathcal{G}_A.$$

To see this, let $p \in P$ with $g(p) = g(p_0), \forall g \in \mathcal{G}_A$. There exist $p' \in P_{A,p_0}$ and $a \in G$ such that $p = p'a$. Due to equivariance, $g(p) = a^{-1}g(p')a$, hence $g(p_0) = a^{-1}g(p_0)a, \forall g \in \mathcal{G}_A$. Thus, a commutes with $\Phi_{p_0}(\mathcal{G}_A)$. Now the stabilizer theorem yields that $a \in \mathrm{C}_G^2\big(H_{A,p_0}\big) = \widetilde{H}_{A,p_0}$, hence $p \in \widetilde{P}_{A,p_0}$.

It follows that $\widetilde{P}_{A,p_0}$ is determined by the subgroup $\mathcal{G}_A$ rather than by A itself. Thus, by assigning $\widetilde{P}_{A,p_0}$ to $\mathcal{G}_A$ we obtain a map from stabilizers to Howe subbundles. Since $\mathcal{G}_A$ can be recovered from $\widetilde{P}_{A,p_0}$ via (9), the map is injective. What kind of Howe subbundles arise in this way from stabilizers? Of course, all of them are generated by a *connected* reduction of P. Howe subbundles with this property will be called holonomy-induced. Conversely, let a holonomy-induced Howe subbundle $\widetilde{Q}$ with generating connected bundle reduction Q be given. As is well known [52], if $\dim M \geqslant 2$, there exist connections in P which have holonomy bundle Q. Then $\widetilde{Q}$ is the Howe subbundle assigned to the stabilizer of any of these connections.

To summarize, we have found, within the C^∞-setting, that stabilizers are in 1–1 correspondence with holonomy-induced Howe subbundles. To carry over this characterization to the conjugacy classes, we note that, for gauge transformations g,

$$P_{A^{(g)},p_0} = \left(\Theta_g\left(P_{A,p_0}\right)\right) g(p_0)^{-1} \tag{10}$$

where Θ_g denotes the vertical automorphism of P defined by g, i.e.

$$\Theta_g(p) = pg(p) \qquad \forall p \in P.$$

Since (10) carries over to the corresponding Howe subbundles, we have to factorize the holonomy-induced Howe subbundles by vertical automorphisms of P. Since any isomorphism of one bundle reduction of P onto another one can be extended to a vertical automorphism of P, the factorization is actually by isomorphy. Moreover, in order to make the construction independent of the chosen point p_0, one must take Howe subbundles modulo the principal action of G on P. Note that then the corresponding structure groups are determined up to conjugacy in G.

Thus, we have found a characterization of the orbit types of the action of smooth local gauge transformations on smooth connections. Finally, one can prove that the action of $\mathcal{G}$ on $\mathcal{C}$ has exactly the same orbit types [65].

Let us summarize.

Theorem 2.3 (Reduction theorem). *The orbit types of the action of $\mathcal{G}$ on $\mathcal{C}$ are in 1–1 correspondence with smooth holonomy-induced Howe subbundles of P modulo isomorphy and modulo the principal action of G on P. The correspondence is given by (9).*

Note that it is obvious from (9) that the partial orderings of orbit types and bundle reductions coincide. For later use, let us introduce the notation $\mathcal{C}^S$ for the subset of connections with stabilizer S, $\mathcal{C}^\sigma$ for the subset of connections of orbit type σ and $\mathcal{M}^\sigma$ for the subset of orbits of type σ. Correspondingly, we define

$$\mathcal{C}^{\leqslant S} := \bigcup_{S' \supseteq S} \mathcal{C}^{S'} \qquad \mathcal{C}^{\leqslant \sigma} := \bigcup_{\sigma' \leqslant \sigma} \mathcal{C}^{\sigma'} \qquad \mathcal{M}^{\leqslant \sigma} := \bigcup_{\sigma' \leqslant \sigma} \mathcal{M}^{\sigma'}$$

and similarly $\mathcal{C}^{\geqslant S}, \mathcal{C}^{\geqslant \sigma}, \mathcal{M}^{\geqslant \sigma}$.

2.4. Decomposition theorem

In what follows we will see that there exists a natural generalization of the Hodge–de Rham decomposition theorem (w.r.t. the L^2-metric γ_0) to the covariant derivatives ∇_A. This has two important consequences. First, it ensures that the orbits of the $\mathcal{G}$-action are submanifolds. Second, it implies that the two distributions on $\mathcal{C}$, defined by

$$\mathfrak{V}_A = \mathrm{im}(\nabla_A) \qquad \mathfrak{H}_A = \ker(\nabla_A^*) \qquad A \in \mathcal{C} \tag{11}$$

provide a natural orthogonal splitting of the tangent bundle

$$\mathrm{T}\mathcal{C} = \mathfrak{V} \oplus \mathfrak{H}. \tag{12}$$

This splitting is fundamental for all constructions discussed within the rest of this and the next three sections. In particular, it is basic for the construction of tubes and slices, it ensures the (locally trivial) fibre bundle structure on each stratum and it induces natural (weak) Riemannian metrics on each stratum of the gauge orbit space via a Kaluza–Klein construction.

Using the theory of differential operators with W^l-coefficients [21, 23], one can verify that the following decompositions hold, see [63] for explicit proofs.

Theorem 2.4 (Decomposition theorem). *Let $A \in \mathcal{C}$. Then*

$$W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P) = \mathrm{im}(\nabla_A) \oplus \ker(\nabla_A^*) \tag{13}$$

$$W^{k-1}(\mathrm{Ad}P) = \mathrm{im}(\Delta_A) \oplus \ker(\Delta_A) \tag{14}$$

where the sums are orthogonal w.r.t. the corresponding L^2-scalar products.

Remarks.

1. The decompositions still hold if one replaces ∇_A, ∇_A^*, Δ_A by $\nabla_A|W^{l+1}$, $\nabla_A^*|W^l$, $\Delta_A|W^{l+1}$, respectively, with $\dim(M)/2 < l \leqslant k$.

2. As an immediate consequence of (13),

$$\ker(\Delta_A) = \ker(\nabla_A) \tag{15}$$
$$\operatorname{im}(\Delta_A) = \operatorname{im}(\nabla_A^*). \tag{16}$$

3. In the decomposition (14), the subspace $\ker(\Delta_A)$ of W^{k+1} is viewed as a subspace of W^{k-1}. Actually, there should occur $\ker(\Delta_A|W^{k-1})$ instead. However, by virtue of point 1 above, formula (15) holds also in degree $\dim(M)/2 < l \leqslant k$. Since d is elliptic, $\ker(\nabla_A|W^{l+1}) = \ker(\nabla_A)$, for any $\dim(M)/2 < l \leqslant k$. Hence, (15) implies $\ker(\Delta_A|W^{k-1}) = \ker(\Delta_A)$.

As an important consequence of the decomposition theorem one has

Theorem 2.5. *For any $A \in \mathcal{C}$, the orbit of A under the action of $\mathcal{G}$ is an embedded submanifold of $\mathcal{C}$, naturally diffeomorphic to $\mathcal{G}/\mathcal{G}_A$.*

This was proved in [54]. Since the orbits are closed due to properness of the action and since the topology of $\mathcal{C}$ is second countable (recall that $\mathcal{C}$ is separable), it suffices to show that the map

$$\iota_A : \mathcal{G} \to \mathcal{C} \qquad g \mapsto A^{(g)} \tag{17}$$

is a subimmersion [14, section 5.12.5]. The map ι_A factors through $\mathcal{G}/\mathcal{G}_A$,

$$\mathcal{G} \to \mathcal{G}/\mathcal{G}_A \xrightarrow{\tilde{\iota}_A} \mathcal{C}.$$

Since the first mapping is the projection in a locally trivial principal bundle, it is a submersion. We claim that $\tilde{\iota}_A$ is a smooth immersion (so that (17) is a subimmersion, indeed).

Smoothness follows from the fact that, due to local triviality of the principal bundle $\mathcal{G} \to \mathcal{G}/\mathcal{G}_A$, $\mathcal{G}/\mathcal{G}_A$ can be covered by smooth local sections $\mathcal{G}/\mathcal{G}_A \supseteq U \to \mathcal{G}$. Namely, locally, $\tilde{\iota}_A$ factors through such a section and ι_A.

To prove that $\tilde{\iota}_A$ is an immersion, it suffices to show that it is an immersion at $[e] \in \mathcal{G}/\mathcal{G}_A$, the class of the identity of $\mathcal{G}$. Given a closed subspace $\mathcal{Y}$ of $\mathrm{L}\mathcal{G}$ complementary to $\mathrm{L}\mathcal{G}_A$, one can find an appropriate local section (U, s) about $[e]$ such that its tangent map $(s_*)_{[e]}$ maps $\mathrm{T}_{[e]}\mathcal{G}/\mathcal{G}_A$ isomorphically onto $\mathcal{Y}$. Then

$$(\tilde{\iota}_{A*})_{[e]} \circ \left((s_*)_{[e]}\right)^{-1} = (\iota_{A*})_e|_{\mathcal{Y}}.$$

Since $(s_*)_{[e]}$ is an isomorphism, it suffices to show that the rhs is injective and has closed image. For that purpose, recall that the Killing field at A generated by ξ is

$$\iota_{A*}\xi = \nabla_A \xi. \tag{18}$$

Now, injectivity is obvious from (8). Moreover, $\operatorname{im}((\iota_{A*})_e|_{\mathcal{Y}}) = \operatorname{im}(\nabla_A)$ and, due to the decomposition theorem, the image is closed and admits a closed complement.

As a second important consequence of the decomposition theorem we note that the tangent bundle splitting (12) holds and is orthogonal w.r.t. the L^2-metric γ^0. Due to (6), the distributions $\mathfrak{V}$ and $\mathfrak{H}$ are equivariant,

$$\mathfrak{V}_{A^{(g)}} = (\mathfrak{V}_A)^{(g)} \qquad \mathfrak{H}_{A^{(g)}} = (\mathfrak{H}_A)^{(g)}. \tag{19}$$

Geometrically, $\mathfrak{V}$ consist of the subspaces tangent to the orbits. We stress that, in general, neither $\mathfrak{V}$ nor $\mathfrak{H}$ are smooth or locally trivial. However, as we will see later, restrictions to strata will be so.

Let us determine the projectors

$$\mathbf{v}, \mathbf{h} : \mathrm{T}\mathcal{C} \to \mathrm{T}\mathcal{C}$$

onto $\mathfrak{V}$ and $\mathfrak{H}$, respectively. They are given by maps

$$\mathcal{C} \to \mathrm{B}(\mathcal{T}) \qquad A \mapsto \mathbf{v}_A, \mathbf{h}_A$$

where $\mathbf{v}_A$ and $\mathbf{h}_A$ denote the projectors associated with the decomposition (13). Since $\ker(\Delta_A) \subseteq W^{k+1}$, the decomposition (14) implies

$$W^{k+1} = \ker(\Delta_A) \oplus \ker(\Delta_A)^{\perp_0} \tag{20}$$

where $\ker(\Delta_A)^{\perp_0} = W^{k+1} \cap \mathrm{im}(\Delta_A)$. Thus, by restriction, Δ_A induces a bounded operator $\ker(\Delta_A)^{\perp_0} \to \mathrm{im}(\Delta_A)$ which is invertible, hence has bounded inverse by the open mapping theorem. The inverse can be prolonged to a bounded operator

$$\mathrm{G}_A : W^{k-1}(\mathrm{Ad}P) \to W^{k+1}(\mathrm{Ad}P) \tag{21}$$

the Green's operator associated with Δ_A, by setting $\mathrm{G}_A|\ker(\Delta_A) = 0$. Note that $\mathrm{G}_A\Delta_A : W^{k+1} \to W^{k+1}$ is the L^2-orthogonal projector onto $\ker(\Delta_A)^{\perp_0}$. Hence,

$$\nabla_A \mathrm{G}_A \Delta_A = \nabla_A \qquad \Delta_A \mathrm{G}_A \nabla_A^* = \nabla_A^*. \tag{22}$$

Note, in particular, that G_A is not the inverse of Δ_A, unless $\mathcal{G}_A$ is discrete, as in the case of the principal stratum for semisimple structure group [60].

Now consider the composition $\nabla_A \mathrm{G}_A \nabla_A^*$, which is a bounded operator on $\mathcal{T}$. Using (22) one can check that it is a projector and that it acts trivially on $\mathfrak{H}_A$ and identically on $\mathfrak{V}_A$. Thus

$$\mathbf{v}_A = \nabla_A \mathrm{G}_A \nabla_A^* \qquad \mathbf{h}_A = 1 - \mathbf{v}_A. \tag{23}$$

From (6) we infer

$$\mathrm{G}_{A^{(g)}} = \mathrm{Ad}(g^{-1})\, \mathrm{G}_A\, \mathrm{Ad}(g). \tag{24}$$

It follows

$$\mathbf{v}_{A^{(g)}} = \mathrm{Ad}(g^{-1})\, \mathbf{v}_A\, \mathrm{Ad}(g) \qquad \mathbf{h}_{A^{(g)}} = \mathrm{Ad}(g^{-1})\, \mathbf{h}_A\, \mathrm{Ad}(g) \tag{25}$$

which is consistent with (19).

2.5. Slice theorem

We assume the reader to be familiar with the notions of tube and slice [19]. They are generalizations of the notions of local trivialization and local section, respectively, which apply to group actions with a single orbit type.

Following [54], the normal distribution $\mathfrak{H}$ can be used to construct tubes and slices for the action of $\mathcal{G}$ on $\mathcal{C}$. For $x \in \mathcal{M}$, the normal bundle of the orbit $\pi^{-1}(x)$ is given by

$$N_x := \mathfrak{H}|_{\pi^{-1}(x)}.$$

According to (19), N_x is equivariant. We claim that it is a smooth locally trivial vector subbundle of $\mathrm{T}\mathcal{C}|_{\pi^{-1}(x)}$. To see this, observe that for given $A \in \pi^{-1}(x)$, due to local triviality of the principal bundle $\mathcal{G} \to \mathcal{G}/\mathcal{G}_A$, there exists a neighbourhood U_A of A in $\pi^{-1}(x)$ and a smooth map $\theta : U_A \to \mathcal{G}$ such that $A' = A^{(\theta(A'))}$, for any $A' \in U_A$. The map

$$U_A \times \mathcal{T} \to \mathrm{T}\mathcal{C}|_{U_A} \qquad (A', X) \mapsto \left(A', X^{(\theta(A'))}\right)$$

is easily seen to be a diffeomorphism. Due to equivariance of N_x, the pre-image of $N_x|_{U_A}$ under this map is $U_A \times \mathfrak{H}_A$. This proves the assertion. Let us note that the argument shows that any equivariant vector subbundle of $\mathrm{T}\mathcal{C}|_{\pi^{-1}(x)}$ which has closed fibres is smooth and locally trivial.

For $\varepsilon > 0$, define

$$\mathfrak{H}_{A,\varepsilon} := \left\{ X \in \mathfrak{H}_A : \sqrt{\gamma_A^k(X,X)} < \varepsilon \right\}$$

where the W^k-metric γ^k was defined in (4). Consider the smooth subbundle

$$N_{x,\varepsilon} := \{(A, X) \in N_x : X \in \mathfrak{H}_{A,\varepsilon}\}$$

of N_x. Note that $N_{x,\varepsilon}$ is not just the ε-disc bundle of N_x, because orthogonality and length are taken w.r.t. different metrics. Due to $\mathcal{G}$-invariance of γ^k, $N_{x,\varepsilon}$ is equivariant.

As $\mathcal{G}$-spaces, N_x and $N_{x,\varepsilon}$ are equivariantly diffeomorphic through the rescaling map

$$\varrho_\varepsilon : N_x \to N_{x,\varepsilon} \qquad (A, X) \mapsto \left(A, \frac{\varepsilon}{\sqrt{\gamma_A^k(X,X) + 1}} X \right).$$

By restriction, the map

$$\exp : \mathrm{T}\mathcal{C} \to \mathcal{C} \qquad (A, X) \mapsto A + X$$

which is in fact the exponential map w.r.t. the L^2-metric γ^0, defines a smooth $\mathcal{G}$-equivariant map $N_{x,\varepsilon} \to \mathcal{C}$. The image is

$$\mathcal{U}_{x,\varepsilon} = \{A + X : \pi(A) = x, X \in \mathfrak{H}_{A,\varepsilon}\}. \tag{26}$$

It is an open invariant neighbourhood of $\pi^{-1}(x)$ in $\mathcal{C}$ (called 'tubular neighbourhood'). Using that $\pi^{-1}(x)$ is an embedded submanifold, one can show [54] that there exists $\varepsilon > 0$ such that the restriction of exp to $N_{x,\varepsilon} \subseteq \mathrm{T}\mathcal{C}$ is injective. Consequently, the composition

$$\exp \circ \varrho_\varepsilon : N_x \to \mathcal{C} \tag{27}$$

is an equivariant diffeomorphism onto $\mathcal{U}_{x,\varepsilon}$, i.e. it is a tube. (Note that already $\exp|_{N_{x,\varepsilon}}$ alone is a tube.)

From (26) we can easily read off the slice about $A \in \pi^{-1}(x)$ associated with $\mathcal{U}_{x,\varepsilon}$. It is the subset

$$\mathcal{S}_{A,\varepsilon} := \{A + X : X \in \mathfrak{H}_{A,\varepsilon}\}$$

of $\mathcal{U}_{x,\varepsilon}$. By construction, $\mathcal{S}_{A,\varepsilon}$ obeys the defining properties of a slice:

(i) $\mathcal{U}_{x,\varepsilon} = (\mathcal{S}_{A,\varepsilon})^{(\mathcal{G})}$,
(ii) $\mathcal{S}_{A,\varepsilon}$ is closed in $\mathcal{U}_{x,\varepsilon}$,
(iii) $\mathcal{S}_{A,\varepsilon}$ is invariant under the stabilizer $\mathcal{G}_A$,
(iv) For any $g \in \mathcal{G}$, $(\mathcal{S}_{A,\varepsilon})^{(g)} \cap \mathcal{S}_{A,\varepsilon} \neq \emptyset$ implies $g \in \mathcal{G}_A$.

We conclude:

Theorem 2.6 (Slice theorem)**.** *For any $x \in \mathcal{M}$ there exists $\varepsilon > 0$ such that (27) is a tube about x. For any $A \in \mathcal{C}$ there exists $\varepsilon > 0$ such that $\mathcal{S}_{A,\varepsilon}$ is a slice about A. In particular, the action of $\mathcal{G}$ on $\mathcal{C}$ admits a slice at any point.*

In the following, whenever we write $\mathcal{U}_{x,\varepsilon}$ or $\mathcal{S}_{A,\varepsilon}$, it is understood that ε is small enough to make the subset a tubular neighbourhood or a slice, respectively.

The authors of [54] actually prove more: they show that for any $x \in \mathcal{M}$ and any open invariant neighbourhood U of x in $\mathcal{C}$ there exists $\varepsilon > 0$ such that $\overline{\mathcal{U}_{x,\varepsilon}} \subseteq U$ and $U \setminus \overline{\mathcal{U}_{x,\varepsilon}} \neq \emptyset$. They call this the 'local slice theorem'. As a consequence, $\mathcal{M}$ is a regular topological space, meaning that whenever one has a closed subset V and a point $x \notin V$ then there exists a neighbourhood of x, whose closure in $\mathcal{M}$ does not intersect V. According to Urysohn's metrization theorem, regularity in combination with second countability (which is due to separability of $\mathcal{C}$) then implies that $\mathcal{M}$ is a metrizable space.

As an application, let us note an immediate consequence of the slice theorem. Property (iv) of slices implies that for any $x \in \mathcal{M}^\sigma$ and any $A \in \mathcal{C}^S$,

$$\mathcal{U}_{x,\varepsilon} \subseteq \mathcal{C}^{\geqslant\sigma} \qquad \mathcal{S}_{A,\varepsilon} \subseteq \mathcal{C}^{\geqslant S}. \tag{28}$$

It follows that for any stabilizer S and orbit type σ the following subsets are open:

$$\mathcal{C}^S \text{ in } \mathcal{C}^{\leqslant S} \qquad \mathcal{C}^\sigma \text{ in } \mathcal{C}^{\leqslant\sigma} \qquad \mathcal{M}^\sigma \text{ in } \mathcal{M}^{\leqslant\sigma}.$$

To see this, let $A \in \mathcal{C}^S$. Since $\mathcal{U}_{\pi(A),\varepsilon}$ is a neighbourhood of A in $\mathcal{C}$, its intersection with $\mathcal{C}^{\leqslant S}$ is a neighbourhood of A in $\mathcal{C}^{\leqslant S}$. Due to (28), the intersection is contained in

$$\mathcal{C}^{\geqslant S} \cap \mathcal{C}^{\leqslant S} = \mathcal{C}^S.$$

The argument applies without change to $\mathcal{C}^\sigma$. For $\mathcal{M}^\sigma$ it suffices to note that $\mathcal{U}_{x,\varepsilon}$ projects to a neighbourhood of x in $\mathcal{M}$.

2.6. Approximation theorem

It is well known that connections with a trivial stabilizer under $\mathcal{G}$-action are dense in $\mathcal{C}$, see [70]. More generally, the question arises, whether $\mathcal{C}^\sigma$ is dense in $\mathcal{C}^{\leqslant\sigma}$, in other words, whether a connection with a nontrivial stabilizer can be approximated by connections with a prescribed, strictly smaller stabilizer. In [54], the following is proved.

Theorem 2.7 (Approximation theorem)**.** *Assume* $\dim M \geqslant 2$. *Let* $A \in \mathcal{C}$ *and let* Q *be a connected bundle reduction of* P *to a (not necessarily closed) Lie subgroup. Assume that* Q *contains a holonomy bundle of* A. *Then there exists* $X \in \mathcal{T}$ *such that all* $A + tX$, $t \in \mathbb{R}\setminus\{0\}$, *have holonomy bundle* Q.

By virtue of the characterization of stabilizers by bundle reductions of P, see (7), the approximation theorem implies that the following subsets are dense:

$$\mathcal{C}^S \subseteq \mathcal{C}^{\leqslant S} \qquad \mathcal{C}^\sigma \subseteq \mathcal{C}^{\leqslant\sigma} \qquad \mathcal{M}^\sigma \subseteq \mathcal{M}^{\leqslant\sigma}. \tag{29}$$

Namely, let $A \in \mathcal{C}^{\leqslant S}$. Then $S \subseteq \mathcal{G}_A$. Hence, according with (7), the bundle reduction Q_S associated with S, based at some p_0, contains the holonomy bundle of A, based at p_0. Of course, so does already the connected component $Q_{S,p_0} \subseteq Q_S$ of p_0. Thus, theorem 2.7 yields that A can be approximated by connections with holonomy bundle Q_{S,p_0}. By construction, such connections have stabilizer S. Hence, $\mathcal{C}^S$ is dense in $\mathcal{C}^{\leqslant S}$. Then denseness of $\mathcal{C}^\sigma$ in $\mathcal{C}^{\leqslant\sigma}$ and of $\mathcal{M}^\sigma$ in $\mathcal{M}^{\leqslant\sigma}$ follows.

One can combine openness, found above, and denseness by saying that $\mathcal{C}^S, \mathcal{C}^\sigma, \mathcal{M}^\sigma$ are generic sets in $\mathcal{C}^{\leqslant S}, \mathcal{C}^{\leqslant\sigma}$, and $\mathcal{M}^{\leqslant\sigma}$, respectively.

Combining the approximation theorem with the slice theorem one arrives at the following closure formulae: for any orbit type σ,

$$\overline{\mathcal{C}^\sigma} = \mathcal{C}^{\leqslant\sigma} \qquad \overline{\mathcal{M}^\sigma} = \mathcal{M}^{\leqslant\sigma}. \tag{30}$$

Indeed, the inclusions from right to left are obvious from (29). The converse inclusions follow from the slice theorem: let $A \in \overline{\mathcal{C}^\sigma}$. Consider $\mathcal{U}_{A,\varepsilon} \cap \overline{\mathcal{C}^\sigma}$. Since this is a neighbourhood of A in $\overline{\mathcal{C}^\sigma}$, it contains some $B \in \mathcal{C}^\sigma$. According to (28), then $\sigma \geqslant \mathrm{Type}(A)$. Thus, $A \in \mathcal{C}^{\leqslant\sigma}$. The inclusion for $\mathcal{M}^\sigma$ then follows by noting that for saturated sets such as $\mathcal{C}^\sigma$, closure and projection commute.

We remark that for stabilizers S one has a similar formula:

$$\overline{\mathcal{C}^S} = \mathcal{C}^{\leqslant S}. \tag{31}$$

While $\supseteq$ is again due to (29), $\subseteq$ can be proved without the slice theorem by the following simple argument. For any $g \in \mathcal{C}$, consider the map

$$\Phi_g : \mathcal{C} \to \mathcal{T} \qquad A \mapsto A^{(g)} - A.$$

As the Φ_g are continuous, the subsets $\Phi_g^{-1}(0)$ are closed in $\mathcal{C}$. Then $\mathcal{C}^{\leqslant S} = \bigcap_{g\in S} \Phi_g^{-1}(0)$ is closed. Hence, $\overline{\mathcal{C}^S} \subseteq \mathcal{C}^{\leqslant S}$.

3. Smooth fibre bundle structure of strata

In this section, we shall explain how the projections

$$\pi^\sigma : \mathcal{C}^\sigma \to \mathcal{M}^\sigma$$

induced from $\pi : \mathcal{C} \to \mathcal{M}$ can be equipped with the structure of smooth locally trivial fibre bundles. As a result, in a sense, π fibres over the set of orbit types into such bundles.

3.1. Submanifold structure of the configuration space strata

To prove that $\mathcal{C}^\sigma$ is a submanifold of $\mathcal{C}$, it suffices to show that for any $x \in \mathcal{M}^\sigma$ the subset

$$\mathcal{U}^\sigma_{x,\varepsilon} := \mathcal{U}_{x,\varepsilon} \cap \mathcal{C}^\sigma$$

which is a neighbourhood of the orbit $\pi^{-1}(x)$ in $\mathcal{C}^\sigma$, is a submanifold of $\mathcal{U}_{x,\varepsilon}$. For any $A \in \mathcal{C}^\sigma$, define

$$\begin{aligned} \mathcal{S}^\sigma_{A,\varepsilon} &:= \mathcal{S}_{A,\varepsilon} \cap \mathcal{C}^\sigma \\ \mathfrak{H}^\sigma_A &:= \{X \in \mathfrak{H}_A : \mathcal{G}_X \supseteq \mathcal{G}_A\} \\ \mathfrak{H}^\sigma_{A,\varepsilon} &:= \mathfrak{H}_{A,\varepsilon} \cap \mathfrak{H}^\sigma_A. \end{aligned} \tag{32}$$

Due to (28),

$$\mathcal{G}_{A'} = \mathcal{G}_A \qquad \forall A' \in \mathcal{S}^\sigma_{A,\varepsilon}. \tag{33}$$

Hence, $\mathcal{S}^\sigma_{A,\varepsilon} = \{A + X : X \in \mathfrak{H}_{A,\varepsilon}, \mathcal{G}_{A+X} = \mathcal{G}_A\}$. Since $\mathcal{G}_{A+X} = \mathcal{G}_A$ iff $\mathcal{G}_X \supseteq \mathcal{G}_A$,

$$\mathcal{S}^\sigma_{A,\varepsilon} = \left\{A + X : X \in \mathfrak{H}^\sigma_{A,\varepsilon}\right\}. \tag{34}$$

Then

$$\mathcal{U}^\sigma_{x,\varepsilon} = \left\{A + X : A \in \pi^{-1}(x), X \in \mathfrak{H}^\sigma_{A,\varepsilon}\right\}.$$

Therefore, the pre-image of $\mathcal{U}^\sigma_{x,\varepsilon}$ under the equivariant diffeomorphism (27) is the vector subbundle

$$N^\sigma_x := \bigcup_{A\in\pi^{-1}(x)} \mathfrak{H}^\sigma_A$$

of N_x. As we have argued in subsection 2.5, since N^σ_x is equivariant and since its fibres are closed subspaces of $\mathcal{T}$, it is a smooth subbundle of $\mathrm{T}\mathcal{C}|_{\pi^{-1}(x)}$, hence of N_x. It follows that $\mathcal{U}^\sigma_{x,\varepsilon}$ is a smooth submanifold of $\mathcal{U}_{x,\varepsilon}$, for any $x \in \mathcal{M}^\sigma$, as asserted.

For later purposes, let us note that the vector subbundle N^σ_x is in fact trivial, where a smooth trivialization is given by

$$\mathcal{G}/\mathcal{G}_A \times \mathfrak{H}^\sigma_A \to N^\sigma_x \qquad ([g], X) \mapsto \left(A^{(g)}, X^{(g)}\right)$$

for some $A \in \pi^{-1}(x)$. Note that this map is well defined precisely because $\mathcal{G}_X \supseteq \mathcal{G}_A$. It follows that $\mathcal{U}^\sigma_{x,\varepsilon}$ also has a direct product structure. This can be made explicit by introducing maps

$$\chi^\sigma_{A,\varepsilon} : \mathcal{S}^\sigma_{A,\varepsilon} \times \mathcal{G}/\mathcal{G}_A \to \mathcal{U}^\sigma_{\pi(A),\varepsilon} \qquad (A', [g]) \mapsto A'^{(g)} \tag{35}$$

which are easily seen to be diffeomorphisms. Note that, for obvious reasons, the roles of fibre and base have changed here. Also for later purposes, let us note that

$$\mathrm{T}\mathcal{S}^{\sigma}_{A,\varepsilon} = \mathcal{S}^{\sigma}_{A,\varepsilon} \times \mathfrak{H}^{\sigma}_{A} \tag{36}$$

for any $A \in \mathcal{C}^{\sigma}$, which is obvious from (34).

3.2. Manifold structure of the orbit space strata

We shall construct an atlas of the stratum $\mathcal{M}^{\sigma}$ using the partial slices $\mathcal{S}^{\sigma}_{A,\varepsilon}$, $A \in \mathcal{C}^{\sigma}$. For any $x \in \mathcal{M}^{\sigma}$, define

$$V^{\sigma}_{x,\varepsilon} := \pi\left(\mathcal{U}^{\sigma}_{x,\varepsilon}\right).$$

By restriction in domain and range, for any $A \in \pi^{-1}(x)$, π defines a map

$$\pi^{\sigma}_{A,\varepsilon} : \mathcal{S}^{\sigma}_{A,\varepsilon} \to V^{\sigma}_{x,\varepsilon}. \tag{37}$$

We prove:

(i) $\pi^{\sigma}_{A,\varepsilon}$ *is bijective*: Due to (33) and property (iv) of slices, none of the elements of $\mathcal{S}^{\sigma}_{A,\varepsilon}$ has a gauge copy in $\mathcal{S}^{\sigma}_{A,\varepsilon}$.

(ii) $\pi^{\sigma}_{A,\varepsilon}$ *is a homeomorphism onto* $V^{\sigma}_{x,\varepsilon}$: It suffices to check that π maps open subsets of $\mathcal{S}^{\sigma}_{A,\varepsilon}$ to open subsets of $V^{\sigma}_{x,\varepsilon}$. Let $U \subseteq \mathcal{S}^{\sigma}_{A,\varepsilon}$ be open. Then $U = \mathcal{S}^{\sigma}_{A,\varepsilon} \cap U'$, where $U' \subseteq \mathcal{S}_{A,\varepsilon}$ is open. Using a local trivialization of the normal bundle N_x, one can show that the saturation $\tilde{U}' = U'^{(\mathcal{G})}$ is open in $\mathcal{C}$. Since $\mathcal{S}^{\sigma}_{A,\varepsilon}$ does not contain gauge copies, $U = \mathcal{S}^{\sigma}_{A,\varepsilon} \cap \tilde{U}'$. Since $\tilde{U}'$ is saturated,

$$\pi(U) = \pi\left(\mathcal{S}^{\sigma}_{A,\varepsilon}\right) \cap \pi(\tilde{U}') = V^{\sigma}_{x,\varepsilon} \cap \pi(\tilde{U}').$$

Here $\pi(\tilde{U}')$ is open in $\mathcal{M}$. Hence, $\pi(U)$ is open in $V^{\sigma}_{x,\varepsilon}$.

(iii) $V^{\sigma}_{x,\varepsilon}$ *is open in* $\mathcal{M}^{\sigma}$: obviously, $V^{\sigma}_{x,\varepsilon} = \mathcal{M}^{\sigma} \cap \pi(\mathcal{U}_{\pi(A),\varepsilon})$, where $\mathcal{U}_{\pi(A),\varepsilon}$ is open in $\mathcal{C}$.

Since the partial slices $\mathcal{S}^{\sigma}_{A,\varepsilon}$ are open subsets of closed affine subspaces of $\mathcal{C}$, see (34), the family $\left(V^{\sigma}_{\pi(A),\varepsilon}, \left(\pi^{\sigma}_{A,\varepsilon}\right)^{-1}\right)$, $A \in \mathcal{C}^{\sigma}$, provides a covering of $\mathcal{M}^{\sigma}$ by local charts (one can make this more explicit by further mapping $\mathcal{S}^{\sigma}_{A,\varepsilon} \to \mathfrak{H}^{\sigma}_{A,\varepsilon}$). We finally have to check whether the transition maps between these charts are smooth. Due to (35), for $A_1, A_2 \in \mathcal{C}^{\sigma}$ we have a diffeomorphism

$$\mathcal{S}^{\sigma}_{A_1,\varepsilon_1} \cap \mathcal{U}^{\sigma}_{\pi(A_2),\varepsilon_2} \times \mathcal{G}/\mathcal{G}_{A_1} \xrightarrow{\chi^{\sigma}_{A_1,\varepsilon_1}} \mathcal{U}^{\sigma}_{\pi(A_1),\varepsilon_1} \cap \mathcal{U}^{\sigma}_{\pi(A_2),\varepsilon_2} \xrightarrow{(\chi^{\sigma}_{A_2,\varepsilon_2})^{-1}} \mathcal{S}^{\sigma}_{A_2,\varepsilon_2} \cap \mathcal{U}^{\sigma}_{\pi(A_1),\varepsilon_1} \times \mathcal{G}/\mathcal{G}_{A_2}.$$

The transition map $\left(\pi^{\sigma}_{A_2,\varepsilon_2}\right)^{-1} \circ \pi^{\sigma}_{A_1,\varepsilon_1}$ is given by the composition of the embedding $A' \mapsto (A', [e])$, the above diffeomorphism, and projection to the first component. Hence, it is smooth. Thus, the atlas we have constructed equips $\mathcal{M}^{\sigma}$ with the structure of a smooth Hilbert manifold.

3.3. Smooth fibre bundle structure

Using the local diffeomorphisms $\chi^{\sigma}_{A,\varepsilon}$, we obtain local diffeomorphisms

$$V^{\sigma}_{\pi(A),\varepsilon} \times \mathcal{G}/\mathcal{G}_A \xrightarrow{(\pi^{\sigma}_{A,\varepsilon})^{-1}\times \mathrm{id}} \mathcal{S}^{\sigma}_{\pi(A),\varepsilon} \times \mathcal{G}/\mathcal{G}_A \xrightarrow{\chi^{\sigma}_{A,\varepsilon}} \mathcal{U}^{\sigma}_{\pi(A),\varepsilon}$$

which provide a covering of $\mathcal{C}^{\sigma}$ by local trivializations of the projection $\pi^{\sigma} : \mathcal{C}^{\sigma} \to \mathcal{M}^{\sigma}$. Thus, the latter is a smooth locally trivial fibre bundle with standard fibre $\mathcal{G}/\mathcal{G}_A$, for some $A \in \mathcal{C}^{\sigma}$. In particular, π^{σ} is a submersion, because locally it is the projection onto the first component.

Let us consider, in particular, the principal orbit type $\sigma = \mathrm{p}$, which is the conjugacy class consisting of the subgroup $\tilde{\mathrm{Z}}(G)$ of constant functions $P \to \mathrm{Z}(G)$, where $\mathrm{Z}(G)$ denotes the centre of G. Since $\tilde{\mathrm{Z}}(G)$ is normal in $\mathcal{G}$, the smooth locally trivial fibre bundle

$$\pi^{\mathrm{p}} : \mathcal{C}^{\mathrm{p}} \to \mathcal{M}^{\mathrm{p}} \tag{38}$$

is in fact principal, with structure group $\widetilde{\mathcal{G}} := \mathcal{G}/\tilde{\mathrm{Z}}(G)$. This bundle has been studied intensively [58–60, 70]. An important aspect is that nontriviality of this bundle is an obstruction to the existence of smooth (or even continuous) gauges. An elegant argument to show nontriviality, i.e. nonexistence of smooth gauges, is due to Singer [70]. Namely, assume that the bundle was trivial, i.e. $\mathcal{C}^{\mathrm{p}} \cong \mathcal{M}^{\mathrm{p}} \times \widetilde{\mathcal{G}}$. Since $\mathcal{C}^{\mathrm{p}}$ is contractible, then the homotopy groups were $\pi_i(\widetilde{\mathcal{G}}) = 0$, $i \geqslant 1$. Since in many cases this is not true, one concludes that in these cases (38) is nontrivial. For $G = \mathrm{SU}(n)$, examples of this situation are: spacetime manifolds $M = \mathrm{S}^3$ and S^4 [70], T^4 and $\mathrm{S}^2 \times \mathrm{S}^2$ [51] and others. This explains the Gribov ambiguity [40] for the corresponding models.

Remark. For the other orbit types, representatives S are not normal in $\mathcal{G}$. In order to have a similar picture as in the case of the principal stratum, one would have to take the submanifold $\mathcal{C}^S$ of connections with stabilizer S. $\mathcal{C}^S$ is acted upon freely by N/S, where N denotes the normalizer of S in $\mathcal{G}$. Provided one could show that N is a Lie subgroup of $\mathcal{G}$—a problem which, to our knowledge, is not settled yet—the projection $\pi^S : \mathcal{C}^S \to \mathcal{M}^\sigma$ would be a smooth locally trivial principal fibre bundle and $\pi^\sigma : \mathcal{C}^\sigma \to \mathcal{M}^\sigma$ would be associated with this bundle via the action of N/S on $\mathcal{G}/S$.

4. The stratification of the gauge orbit space $\mathcal{M}$

A stratification of a topological space X is a countable disjoint decomposition into smooth manifolds X_i, $i \in I$ (so-called strata), such that the ‘frontier condition‘ is satisfied:

$$X_i \cap \overline{X_{i'}} \neq \emptyset \quad \Rightarrow \quad X_i \subseteq \overline{X_{i'}} \qquad \forall i, i' \in I.$$

As this notion is rather weak, one usually adds additional assumptions about the linking between the strata, thus arriving at special types of stratification. According to [54], the type of stratification appropriate for our purposes is called ‘regular’ and is defined by the property

$$X_i \cap \overline{X_{i'}} \neq \emptyset \quad \Rightarrow \quad X_i \text{ closed in } X_i \cup X_{i'} \qquad \forall i, i' \in I.$$

The following is due to Kondracki and Rogulski [54].

Theorem 4.1 (Stratification theorem)**.** *The decomposition of $\mathcal{M}$ by orbit types is a regular stratification.*

To prove it, one has to check countability of orbit types and the frontier and regularity conditions.

4.1. Countability of orbit types

Due to the reduction theorem, orbit types are in 1–1 correspondence with certain reductions of P to Howe subgroups, modulo isomorphy of the reductions and modulo conjugacy of the subgroups. We note the following facts:

(i) Howe subgroups are closed.
(ii) There are at most countably many conjugacy classes of closed subgroups in a compact group [52].

(iii) There are at most countably many isomorphism classes of principal bundles with a given structure group over a compact manifold. The classes are in 1–1 correspondence with arch-wise connected components of the space of continuous maps from the base space of the bundle to the classifying space. General arguments ensure that there are at most countably many such components.

It follows from (i)–(iii) that the number of orbit types is at most countable.

Let us note that the number of Howe subgroups in a compact Lie group is actually finite. This follows from the fact that any centralizer in a compact Lie group is generated by finitely many elements [14, chapter 9] and that a compact group action on a compact manifold has a finite number of orbit types [19].

4.2. Frontier and regularity conditions

Let σ, σ' be orbit types such that $\overline{\mathcal{M}^\sigma} \cap \mathcal{M}^{\sigma'} \neq \emptyset$. According to the closure formula (30), $\overline{\mathcal{M}^\sigma}$ is a union of strata. If $\mathcal{M}^{\sigma'}$ intersects the union, it must in fact coincide with one of these strata. Then $\mathcal{M}^{\sigma'} \subseteq \overline{\mathcal{M}^\sigma}$. Thus, the decomposition by orbit types satisfies the frontier condition.

On the other hand, we know from the slice theorem that $\mathcal{M}^\sigma$ is open in $\mathcal{M}^{\leqslant\sigma}$, hence in $\overline{\mathcal{M}^\sigma}$. Then $\mathcal{M}^\sigma$ is open in $\mathcal{M}^\sigma \cup \mathcal{M}^{\sigma'}$, because the latter is a subset of $\overline{\mathcal{M}^\sigma}$ due to the frontier condition. Then $\mathcal{M}^{\sigma'}$, being the complement, is closed. Hence, the decomposition by orbit types is a regular stratification.

(This actually shows that if all strata are open in their closures, i.e., locally closed, the frontier condition implies regularity.)

Remarks.

1. Consider the relation
$$\mathcal{M}^\sigma \leqslant \mathcal{M}^{\sigma'} \quad \Leftrightarrow \quad \overline{\mathcal{M}^\sigma} \cap \mathcal{M}^{\sigma'} \neq \emptyset.$$
For any stratification, this relation is reflexive and transitive, i.e., a quasi-ordering (the 'natural quasi-ordering' of the stratification). If the stratification is regular, the relation is also antisymmetric, hence a partial ordering. As for the stratification of $\mathcal{M}$ by orbit types, (30) implies that the natural partial ordering of the strata is just inverse to that of the corresponding orbit types.
2. Instead of using Sobolev techniques one can also stick to smooth connection forms and gauge transformations. Then one obtains essentially analogous results about the stratification of the corresponding gauge orbit space where, roughly speaking, one has to replace 'Hilbert manifold' and 'Hilbert Lie group' by 'tame Fréchet manifold' and 'tame Fréchet Lie group', see [1, 2].

5. L^2-Riemannian structure on strata

The L^2-metric γ^0 on $\mathcal{C}$ induces a weak Riemannian metric on each stratum $\mathcal{M}^\sigma$. This was discussed for the case of the principal stratum in [11, 71] and for the general case in [12]. The basic idea consists in restricting the tangent bundle splitting (12) to strata. This yields a smooth connection in each bundle which allows tangent vectors to be lifted, thus projecting γ^0 to a metric on each stratum.

5.1. A natural connection

By restriction, the distribution $\mathfrak{V}$, made up by the tangent spaces of the orbits, induces a distribution $\mathfrak{V}^\sigma$ on $\mathcal{C}^\sigma$. In contrast to $\mathfrak{V}, \mathfrak{V}^\sigma$ is smooth and locally trivial, because $\mathfrak{V}^\sigma = \ker(\pi^\sigma{}_*)$ and π^σ is a smooth submersion. Let $\mathfrak{H}^\sigma$ denote the normal distribution associated with $\mathfrak{V}^\sigma$ w.r.t. the L^2-metric γ^0. By construction

$$\mathfrak{H}^\sigma := \mathfrak{H} \cap \mathrm{T}\mathcal{C}^\sigma.$$

Due to (12) and $\mathfrak{V}^\sigma \subseteq \mathrm{T}\mathcal{C}^\sigma$,

$$\mathrm{T}\mathcal{C}^\sigma = \mathfrak{V}^\sigma \oplus \mathfrak{H}^\sigma \tag{39}$$

where the sum is orthogonal w.r.t. γ^0. Moreover, $\mathfrak{H}^\sigma$ is $\mathcal{G}$-equivariant

$$\mathfrak{H}^\sigma_{A^{(g)}} = \left(\mathfrak{H}^\sigma_A\right)^{(g)}.$$

We draw the attention of the reader to the fact that we had already introduced the notation $\mathfrak{H}^\sigma_A$ for the subspace of $\mathfrak{H}_A$ consisting of elements invariant under $\mathcal{G}_A$, see (32). This notation suggests that $\mathfrak{H}^\sigma_A$ is in fact the fibre at A of the distribution $\mathfrak{H}^\sigma$. To see that this holds indeed, recall that $\mathfrak{H}_A = \mathrm{T}_A\mathcal{S}_{A,\varepsilon}$. Hence, the fibre of $\mathfrak{H}^\sigma$ is

$$\mathrm{T}_A\mathcal{S}_{A,\varepsilon} \cap \mathrm{T}_A\mathcal{C}^\sigma = \mathrm{T}_A\mathcal{S}^\sigma_{A,\varepsilon}.$$

According to (36), the rhs is given by $\mathfrak{H}^\sigma_A$.

In the remaining part of this subsection we shall prove that the distribution $\mathfrak{H}^\sigma$ is smooth and locally trivial (viewed as a subbundle of $\mathrm{T}\mathcal{C}^\sigma$). Note that, due to weakness of γ^0, this is not obvious from smoothness and local triviality of $\mathfrak{V}^\sigma$. It follows then that $\mathfrak{H}^\sigma$ is a smooth connection in the $\mathcal{G}$-bundle $\pi^\sigma : \mathcal{C}^\sigma \to \mathcal{M}^\sigma$.

Smoothness of $\mathfrak{H}^\sigma$ would follow from smoothness of either one of the corresponding γ^0-orthogonal projectors $\mathbf{h}|_{\mathrm{T}\mathcal{C}^\sigma}$ or $\mathbf{v}|_{\mathrm{T}\mathcal{C}^\sigma}$ which, in turn, would follow from smoothness of the restrictions of $\mathbf{h}$ or $\mathbf{v}$, respectively, to $\mathrm{T}\mathcal{C}|_{\mathcal{C}^\sigma}$. Recall from (23) that the restriction of $\mathbf{v}$ is given by the map

$$\mathcal{C}^\sigma \to \mathrm{B}(\mathcal{T}) \qquad A \mapsto \nabla_A \mathrm{G}_A \nabla_A^*.$$

This map decomposes as

$$\mathcal{C}^\sigma \xrightarrow{\text{diag}} \mathcal{C}^\sigma \times \mathcal{C}^\sigma \times \mathcal{C}^\sigma \xrightarrow{\nabla_\cdot \times \mathrm{G}_\cdot \times \nabla_\cdot^*} \mathrm{B}(W^{k+1}, W^k) \times \mathrm{B}(W^{k-1}, W^{k+1}) \times \mathrm{B}(W^k, W^{k-1}) \xrightarrow{\text{comp.}} \mathrm{B}(W^k).$$

Since diagonal embedding, $\nabla_\cdot$, $\nabla_\cdot^*$ and composition of bounded operators are continuous (multi-) linear maps, it suffices to prove smoothness of the map

$$\mathcal{C}^\sigma \to \mathrm{B}(W^{k-1}, W^{k+1}) \qquad A \mapsto G_A. \tag{40}$$

Pulling it back with a local trivialization $\chi^\sigma_{A_0,\varepsilon}$, $A_0 \in \mathcal{C}^\sigma$, see (35), we obtain a map

$$\mathcal{S}^\sigma_{A_0,\varepsilon} \times \mathcal{G}/\mathcal{G}_{A_0} \to \mathrm{B}(W^{k-1}, W^{k+1}) \qquad (A, [g]) \mapsto G_{A^{(g)}}$$

which is well defined, because $\mathcal{G}_A = \mathcal{G}_{A_0}$, $\forall A \in \mathcal{S}^\sigma_{A_0,\varepsilon}$. Due to (24), this map is smooth along $\mathcal{G}/\mathcal{G}_{A_0}$. Thus, what we actually have to show is that the restrictions of the map (40) to the partial slices $\mathcal{S}^\sigma_{A_0,\varepsilon}$, $A_0 \in \mathcal{C}^\sigma$, are smooth. For that purpose, recall that G_A is constructed from the (bounded) inverse of the operator

$$\widetilde{\Delta}_A : \ker(\Delta_A)^{\perp_0} \to \mathrm{im}(\Delta_A) \tag{41}$$

induced by Δ_A. Due to $\mathcal{G}_A = \mathcal{G}_{A_0}$, equation (15) and the decomposition theorem, we have

$$\ker(\Delta_A) = \ker\left(\Delta_{A_0}\right) \qquad \mathrm{im}(\Delta_A) = \mathrm{im}\left(\Delta_{A_0}\right). \tag{42}$$

Hence, (41) reads

$$\widetilde{\Delta}_A : \ker\left(\Delta_{A_0}\right)^{\perp_0} \to \operatorname{im}\left(\Delta_{A_0}\right) \qquad \forall A \in \mathcal{S}^\sigma_{A_0,\varepsilon}.$$

Thus, the map under consideration decomposes into

$$\mathcal{S}^\sigma_{A_0,\varepsilon} \xrightarrow{\widetilde{\Delta}_\cdot} \operatorname{Inv}\left(\ker\left(\Delta_{A_0}\right)^{\perp_0}, \operatorname{im}\left(\Delta_{A_0}\right)\right) \xrightarrow{\text{inv}} \operatorname{Inv}\left(\operatorname{im}\left(\Delta_{A_0}\right), \ker\left(\Delta_{A_0}\right)^{\perp_0}\right)$$

followed by prolongation to a bounded operator $W^{k-1} \to W^{k+1}$. Here $\operatorname{Inv}(\cdot,\cdot) \subseteq \mathrm{B}(\cdot,\cdot)$ denotes the open subset of invertible bounded operators, whereas 'inv' stands for the inversion map, which is smooth. Since the first step factorizes into continuous linear maps and composition of bounded operators, it is smooth, too.

This concludes the proof of smoothness of the projectors $\mathbf{v}|_{\mathrm{T}\mathcal{C}^\sigma}$ and $\mathbf{h}|_{\mathrm{T}\mathcal{C}^\sigma}$ and, hence, of the distribution $\mathfrak{H}^\sigma$.

Next, let us construct local trivializations of $\mathfrak{H}^\sigma$. To this end, for $A_0 \in \mathcal{C}^\sigma$, consider the distribution $\mathfrak{D}^\sigma_{A_0,\varepsilon}$ on $\mathcal{S}^\sigma_{A_0,\varepsilon} \times \mathcal{G}/\mathcal{G}_{A_0}$, made up by the subspaces tangent to $\mathcal{S}^\sigma_{A_0,\varepsilon}$. Due to (36), it is trivial. We claim that the map

$$\mathfrak{D}^\sigma_{A_0,\varepsilon} \to \mathrm{T}\left(\mathcal{S}^\sigma_{A_0,\varepsilon} \times \mathcal{G}/\mathcal{G}_{A_0}\right) \overset{(\chi^\sigma_{A_0,\varepsilon})_*}{\to} \mathrm{T}\mathcal{U}^\sigma_{\pi(A_0),\varepsilon} \overset{\mathbf{h}}{\to} \mathfrak{H}^\sigma\big|_{\mathcal{U}^\sigma_{A_0,\varepsilon}} \tag{43}$$

is a smooth vector bundle isomorphism and, thus, provides a local trivialization of $\mathfrak{H}^\sigma$. To see this, note that $\left(\chi^\sigma_{A_0,\varepsilon}\right)_*$ maps $\mathfrak{D}^\sigma_{A_0,\varepsilon}$ isomorphically on the equivariant distribution

$$\bigcup_{[g]\in\mathcal{G}/\mathcal{G}_{A_0}} \mathrm{T}\mathcal{S}^\sigma_{A_0^{(g)},\varepsilon}.$$

Hence, due to equivariance of $\mathfrak{H}^\sigma$ and $\mathbf{h}$, it suffices to show that the map

$$\mathrm{T}\mathcal{S}^\sigma_{A_0,\varepsilon} \overset{\mathbf{h}}{\to} \mathfrak{H}^\sigma\big|_{\mathcal{S}^\sigma_{A_0,\varepsilon}} \tag{44}$$

is a smooth vector bundle isomorphism. We shall construct a smooth inverse.

Recall that $\mathcal{S}_{A_0,\varepsilon}$ is transversal to any orbit it meets. Hence,

$$\mathfrak{H}_{A_0} \cap \mathfrak{V}_A = \ker\left(\nabla^*_{A_0}\right) \cap \operatorname{im}(\nabla_A) = \{0\} \qquad \forall A \in \mathcal{S}^\sigma_{A_0,\varepsilon}.$$

Then $\Delta_{A_0 A} := \nabla^*_{A_0}\nabla_A$ has kernel $\ker(\nabla_A) = \ker(\Delta_A)$ and image $\operatorname{im}(\nabla^*_{A_0}) = \operatorname{im}(\Delta_{A_0})$. In particular, for any element A of the partial slice $\mathcal{S}^\sigma_{A_0,\varepsilon}$, $\ker(\Delta_{A_0A}) = \ker(\Delta_{A_0})$. Thus, we can construct a partial inverse G_{A_0A} similar to G_{A_0} and G_A. By construction

$$\mathrm{G}_{A_0A}\Delta_{A_0A} = \mathrm{G}_{A_0}\Delta_{A_0} = \mathrm{G}_A\Delta_A \qquad \Delta_{A_0A}\mathrm{G}_{A_0A} = \Delta_{A_0}\mathrm{G}_{A_0} = \Delta_A\mathrm{G}_A. \tag{45}$$

Define $\mathbf{h}_{A_0A} := \mathrm{id}_{\mathcal{T}} - \nabla_A\mathrm{G}_{A_0A}\nabla^*_A$. Using (22) and (45), one can check that

$$\mathbf{h}_{A_0A}\mathbf{h}_A = \mathbf{h}_{A_0}\mathbf{h}_{A_0A} = \mathbf{h}_{A_0A} \qquad \mathbf{h}_{A_0A}\mathbf{h}_{A_0} = \mathbf{h}_{A_0} \qquad \mathbf{h}_A\mathbf{h}_{A_0A} = \mathbf{h}_A \tag{46}$$

for any $A \in \mathcal{S}^\sigma_{A_0,\varepsilon}$. It follows that $\mathbf{h}_{A_0A}$ maps $\mathfrak{H}_A$ to $\mathfrak{H}_{A_0}$. Since, due to (6),

$$\mathbf{h}_{A_0^{(g)}A^{(g)}} = \operatorname{Ad}(g^{-1})\mathbf{h}_{A_0A}\operatorname{Ad}(g)$$

$\mathbf{h}_{A_0A}$ maps $\mathfrak{H}^\sigma_A$ onto $\mathfrak{H}^\sigma_{A_0}$. Formulae (46) imply

$$\mathbf{h}_{A_0A}\mathbf{h}_A\big|\mathfrak{H}^\sigma_{A_0} = \mathrm{id}_{\mathfrak{H}^\sigma_{A_0}} \qquad \mathbf{h}_A\mathbf{h}_{A_0A}\big|\mathfrak{H}^\sigma_A = \mathrm{id}_{\mathfrak{H}^\sigma_A} \qquad \forall A \in \mathcal{S}^\sigma_{A_0A}.$$

Since the map $\mathcal{S}^\sigma_{A_0A} \to \mathrm{B}(\mathcal{T})$, $A \mapsto \mathbf{h}_{A_0A}$, is smooth, which can be shown in a similar way as for the map $A \mapsto \mathbf{h}_A$, it provides the desired inverse of (44), thus proving that (43) is a local trivialization of $\mathfrak{H}^\sigma$.

We remark that the operators $\mathbf{h}_{A_0A}$ and $\mathbf{v}_{A_0A} := \nabla_A\mathrm{G}_{A_0A}\nabla^*_{A_0}$, where $A \in \mathcal{S}^\sigma_{A_0A}$, $A_0 \in \mathcal{C}^\sigma$, are the projectors associated with the (not necessarily L^2-orthogonal) decomposition

$$\mathcal{T} = \mathfrak{V}_A \oplus \mathfrak{H}_{A_0}.$$

This can be checked using (22) again.

Finally, we note that, with the above connection, there is an associated equivariant differential form with values in $\mathrm{L}\mathcal{G}$, given by

$$\Omega_A(A, X) := \mathrm{G}_A \nabla_A^* X, \tag{47}$$

for all $(A, X) \in \mathcal{C} \times \mathcal{T} = \mathrm{T}\mathcal{C}$. For the principal stratum $\mathcal{M}^{\mathrm{p}}$, we have

$$\Omega_A(A, \nabla_A \xi) = \xi \qquad \forall \xi \in \mathrm{L}\mathcal{G} \tag{48}$$

showing that Ω is an ordinary connection form in the principal fibre bundle over $\mathcal{M}^{\mathrm{p}}$ with structure group $\mathcal{G}$ factorized by its centre. For the other strata, however, Ω_A maps the Killing field generated by ξ to the projection of ξ onto the L^2-orthogonal complement of $\mathrm{L}\mathcal{G}_A$ in $\mathrm{L}\mathcal{G}$. We further comment on this below.

5.2. *The metric*

The natural connection $\mathfrak{H}^\sigma$ and the Riemannian metric γ^0 induce a Riemannian metric $\gamma^{0,\sigma}$ on $\mathcal{M}^\sigma$ as follows. Due to the open mapping theorem, restriction of ${\pi^\sigma}_*$ to a fibre $\mathfrak{H}^\sigma_A$, $A \in \mathcal{C}^\sigma$, induces a Banach space isomorphism onto $\mathrm{T}_{\pi(A)}\mathcal{M}^\sigma$. This allows tangent vectors at $x \in \mathcal{M}^\sigma$ to be lifted to horizontal tangent vectors at $A \in \pi^{-1}(x)$, evaluating their scalar product w.r.t. γ^0. Due to equivariance of $\mathfrak{H}^\sigma$ and invariance of γ^0, the result does not depend on the choice of the representative A. Due to smoothness of $\mathfrak{H}^\sigma$, the Riemannian metric $\gamma^{0,\sigma}$ on $\mathcal{M}^\sigma$ so constructed is smooth.

Let us determine the local representatives of $\gamma^{0,\sigma}$ w.r.t. the charts $\left(\pi^\sigma_{A_0,\varepsilon}\right)^{-1}$, $A_0 \in \mathcal{C}^\sigma$, see (37). Let $A \in \mathcal{S}^\sigma_{A_0,\varepsilon}$. For tangent vectors $(A, X_i) \in \mathrm{T}_A\mathcal{S}^\sigma_{A_0,\varepsilon} = \mathcal{S}^\sigma_{A_0,\varepsilon} \times \mathfrak{H}^\sigma_{A_0}$, we have

$$\left(\pi^\sigma_{A_0,\varepsilon}\right)^* \gamma^{0,\sigma}((A, X_1), (A, X_2)) = \gamma^{0,\sigma}\left(\left(\pi^\sigma_{A_0,\varepsilon}\right)_* (A, X_1), \left(\pi^\sigma_{A_0,\varepsilon}\right)_* (A, X_2)\right).$$

Horizontal lift of $\left(\pi^\sigma_{A_0,\varepsilon}\right)_*(A, X_i)$ to A yields $(A, \mathbf{h}_A X_i)$. Hence

$$\left(\pi^\sigma_{A_0,\varepsilon}\right)^* \gamma^{0,\sigma}((A, X_1), (A, X_2)) = (X_1, \mathbf{h}_A X_2)_0 \tag{49}$$

where we have used $\mathbf{h}_A^* = \mathbf{h}_A$ and $\mathbf{h}_A^2 = \mathbf{h}_A$. In this formula, we can replace $\mathbf{h}_A$ by $\mathbf{h}_{A_0}\mathbf{h}_A$. Since the latter maps $\mathfrak{H}^\sigma_{A_0}$ to itself, the operator which represents the scalar product (49) on $\mathfrak{H}^\sigma_{A_0}$ is

$$\mathbf{h}_{A_0}\mathbf{h}_A\big|_{\mathfrak{H}^\sigma_{A_0}}.$$

Thus, w.r.t. the charts $\left(\pi^\sigma_{A_0,\varepsilon}\right)^{-1}$, $\gamma^{0,\sigma}$ is given by the smooth map

$$\mathcal{S}^\sigma_{A_0,\varepsilon} \to \mathrm{B}\left(\mathfrak{H}^\sigma_{A_0}\right) \qquad A \mapsto \mathbf{h}_{A_0}\mathbf{h}_A\big|_{\mathfrak{H}^\sigma_{A_0}}.$$

Using (46), one can check that the inverse of $\mathbf{h}_{A_0}\mathbf{h}_A\big|_{\mathfrak{H}^\sigma_{A_0}}$ is given by $\mathbf{h}_{A_0A}\mathbf{h}^*_{A_0A}$. In particular, $\mathbf{h}^\sigma_{A_0}\mathbf{h}^\sigma_A\big|_{\mathfrak{H}^\sigma_{A_0}}$ is indeed a Banach space isomorphism.

Remarks.

1. It can be easily seen that the $\mathcal{G}$-invariant L^2-metric γ^0 on the bundle space $\mathcal{C}^\sigma$ is uniquely characterized by the triple $(\gamma^{0,\sigma}, \Omega, (\cdot,\cdot)_0)$, where $(\cdot,\cdot)_0$ denotes the L^2-scalar product on $\mathrm{L}\mathcal{G}$. This is a structure similar to that in Kaluza–Klein theory, where G-invariant metrics η on a G-bundle Q with fibre G/H over spacetime M are in 1–1 correspondence with triples $(\eta_M, \omega, \langle\cdot,\cdot\rangle)$. Here η_M is a metric on M, ω is a connection form in the principal bundle P with structure group N/H associated with Q and $\langle\cdot,\cdot\rangle$ is a $\mathrm{Ad}(G)$-invariant scalar product on the Lie algebra of G. Moreover, N denotes the normalizer of H in G. According to the remark at the end of subsection 3.3, in our case it is unclear whether the

normalizer of a given stabilizer $\mathcal{G}_A$ in $\mathcal{G}$ is a Lie subgroup. Thus, we cannot construct the above associated principal bundle and give an interpretation of Ω as a connection form in this bundle.

2. In a similar way one can project W^k-metrics, like γ^k, see (4), or η^k, see (5), to metrics on the strata. To our knowledge this has not been investigated yet, see, however, [42] for results on the restriction of η^2 to some instanton spaces.

5.3. Curvature

The same tedious but straightforward computation as in the case of the principal stratum [11] yields for the local representative of the Riemannian curvature tensor

$$\begin{aligned} &R : \mathcal{S}^\sigma_{A_0,\varepsilon} \to \mathrm{B}\left(\mathfrak{H}^\sigma_{A_0} \otimes \mathfrak{H}^\sigma_{A_0} \otimes \mathfrak{H}^\sigma_{A_0}, \mathfrak{H}^\sigma_{A_0}\right) \\ &R_A(X,Y)Z = \mathbf{h}^\sigma_{A_0}(-2\mathrm{K}_Z\mathrm{G}_A\mathrm{K}^*_X Y - \mathrm{K}_Y\mathrm{G}_A\mathrm{K}^*_X Z + \mathrm{K}_X\mathrm{G}_A\mathrm{K}^*_Y Z) \end{aligned} \tag{50}$$

where $X, Y, Z \in \mathfrak{H}^\sigma_{A_0}$, $A \in \mathcal{S}^\sigma_{A_0,\varepsilon}$ and $\mathrm{K}_X : W^{k+1}(\mathrm{Ad}P) \to W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P)$ denotes taking the commutator with X and $\mathrm{K}^*_X : W^k(\mathrm{T}^*M \otimes \mathrm{Ad}P) \to W^k(\mathrm{Ad}P)$ its formal adjoint.

From (50) one obtains for the local representative of the sectional curvature $\mathfrak{R}$ of a 2-plane $\mathfrak{P} \subseteq \mathfrak{H}^\sigma_{A_0}$

$$\mathfrak{R}_A(\mathfrak{P}) = 3(\mathrm{K}^*_X Y, \mathrm{G}_A\mathrm{K}^*_X Y)_0$$

where $X, Y \in \mathfrak{H}^\sigma_{A_0}$ are orthonormal vectors spanning $\mathfrak{P}$. We claim that the sectional curvature is nonnegative, as in the case of the principal stratum [11, 71]. To see this, denote $\xi = \mathrm{K}^*_X Y$. Since $\xi \in W^{k-1}(\mathrm{Ad}P)$, one can decompose it according to the decomposition theorem $\xi = \xi_{\mathrm{im}} + \xi_{\mathrm{ker}}$. By construction of G_A, $\xi_{\mathrm{im}} = \Delta_A\mathrm{G}_A\xi$ and $\mathrm{im}(\mathrm{G}_A) \perp_0 \ker(\Delta_A)$. It follows

$$(\xi, \mathrm{G}_A\xi)_0 = (\xi_{\mathrm{im}}, \mathrm{G}_A\xi)_0 = (\Delta_A\mathrm{G}_A\xi, \mathrm{G}_A\xi)_0 = (\nabla_A\mathrm{G}_A\xi, \nabla_A\mathrm{G}_A\xi)_0.$$

5.4. Formal volume element

For the case of the principal stratum $\mathcal{M}^{\mathrm{p}}$, a formal expression for the volume element of the metric $\gamma^{0,\mathrm{p}}$ was derived in [10]:

$$\det\left(\mathbf{h}_{A_0}\mathbf{h}_A\big|_{\mathfrak{H}_{A_0}}\right)^{1/2} = \frac{\det\left(\Delta_{A_0A}\right)}{\det\left(\Delta_{A_0}\right)^{1/2}\det(\Delta_A)^{1/2}} \qquad A \in \mathcal{S}^{\mathrm{p}}_{A_0,\varepsilon} \quad A_0 \in \mathcal{C}^{\mathrm{p}} \tag{51}$$

(recall that $\mathfrak{H}^{\mathrm{p}}_{A_0} = \mathfrak{H}_{A_0}$). The function $A \mapsto \det(\Delta_{A_0A})$ is known as the Faddeev–Popov determinant in the background potential A_0. It follows that the functional integral derived by the Faddeev–Popov procedure [30], can be geometrically interpreted as the formal integral defined by the natural L^2-Riemannian structure on $\mathcal{M}^{\mathrm{p}}$ [10]. Schrödinger quantum mechanics on the gauge orbit space has been discussed in this context, see e.g. [39] and references therein.

It is easy to see that (51) extends to the other strata. Namely, for $A_0 \in \mathcal{C}^\sigma$ and $A \in \mathcal{S}^\sigma_{A_0,\varepsilon}$ we have seen that Δ_A, Δ_{A_0}, and Δ_{A_0A} have common kernel $\ker(\Delta_{A_0})$ and image $\mathrm{im}(\Delta_{A_0})$. By defining their determinant as that of the restricted operators

$$\ker\left(\Delta_{A_0}\right)^{\perp_0} \to \mathrm{im}\left(\Delta_{A_0}\right)$$

(i.e. by 'removing zero modes'), one can establish (51) by essentially the same proof as in the case of the principal stratum.

In particular, one can use (51) to formally define an integral for each stratum. However, as for the physical interpretation, the mere sum of such integrals would certainly not be a reasonable extension of the Faddeev–Popov procedure from the principal stratum to the whole orbit space, because it does not take into account any 'interaction' between strata.

5.5. Geodesics

In [12], the following was proved.

Theorem 5.1. *Let* $A \in \mathcal{C}^\sigma$ *and* $X \in \mathfrak{H}_A^\sigma$*. Let I denote the connected component of* 0 *in* $\{t \in \mathbb{R} : A + tX \in \mathcal{C}^\sigma\}$*. Then I is non-empty, open, and*

$$I \to \mathcal{M}^\sigma \qquad t \mapsto \pi^\sigma(A + tX)$$

is a geodesic in $\mathcal{M}^\sigma$*. Conversely, any geodesic in* $\mathcal{M}^\sigma$ *is of this form.*

Note that

$$\nabla_{A+tX}^* X = \nabla_A^* X = 0 \qquad \forall A \in \mathcal{C} \quad X \in \mathfrak{H}_A^\sigma \quad t \in \mathbb{R} \tag{52}$$

so that the straight line $A + tX$ is perpendicular to any orbit it meets. Thus, the theorem says that the geodesics in $\mathcal{M}^\sigma$ are given by projections of segments of straight lines inside $\mathcal{C}^\sigma$ which are perpendicular to orbits.

Note also that the theorem, in particular, shows that the charts $\left(\pi_{A_0,\varepsilon}^\sigma\right)^{-1}$ provide normal coordinates.

In [12], the above characterization of orbits is used to prove that the principal stratum, in general, is not geodesically complete. In fact, the argument given there can be extended to prove

Theorem 5.2. $\mathcal{M}^\sigma$ *is geodesically complete if and only if there does not exist* σ' *such that* $\sigma < \sigma'$*.*

Indeed, for $A \in \mathcal{C}^\sigma$ and $X \in \mathfrak{H}_A^\sigma$, we have $\mathcal{G}_{A+tX} \supseteq \mathcal{G}_A \cap \mathcal{G}_X = \mathcal{G}_A$. Therefore,

$$A + tX \in \mathcal{C}^{\leqslant\sigma} \qquad \forall t \in \mathbb{R}. \tag{53}$$

In particular, if there is no σ' with $\sigma < \sigma'$, the geodesic associated with A and X is defined for all values $t \in \mathbb{R}$.

Now assume that $\sigma < \sigma'$ for some σ'. Choose $x' \in \mathcal{M}^{\sigma'}$ and a tube $\mathcal{U}_{x',\varepsilon}$ about the orbit $\pi^{-1}(x')$. Since $\mathcal{U}_{x',\varepsilon}$ is a neighbourhood of $\pi^{-1}(x')$ in $\mathcal{C}$, the denseness properties (29) imply $\mathcal{U}_{x',\varepsilon} \cap \mathcal{C}^\sigma \neq \emptyset$. Since $\mathcal{U}_{x',\varepsilon} = \bigcup_{A' \in \pi^{-1}(x')} \mathcal{S}_{A',\varepsilon}^{\sigma'}$ one finds A' such that $\mathcal{S}_{A',\varepsilon}^{\sigma'} \cap \mathcal{C}^\sigma \neq \emptyset$. Choose A from the intersection and let $X \in \mathcal{T}$ such that $A' = A + X$. Since $X \in \mathfrak{H}_{A'}^{\sigma'}$, (52) implies that $\nabla_A^* X = 0$. Since $A \in \mathcal{S}_{A',\varepsilon}^{\sigma'}$, $\mathcal{G}_A \subseteq \mathcal{G}_{A'}$. It follows that $X \in \mathfrak{H}_A^\sigma$. Thus, A and X define a geodesic in $\mathcal{M}^\sigma$ that cannot be prolonged to values $t \geqslant 1$.

The following theorem was stated for the principal stratum in [12].

Theorem 5.3. *Let* $A \in \mathcal{C}^\sigma$, $X \in \mathfrak{H}_A^\sigma$*. The set of values* $t \in \mathbb{R}$ *for which* $A + tX \notin \mathcal{C}^\sigma$ *is discrete.*

To see this, denote $C(t) = A + tX$. According to (53), $C^{-1}(\mathcal{C}^\sigma)$ is open in $\mathbb{R}$, because $\mathcal{C}^\sigma$ is open in $\mathcal{C}^{\leqslant\sigma}$. Hence, $\mathbb{R}\backslash C^{-1}(\mathcal{C}^\sigma)$ is closed in $\mathbb{R}$.

Let $t_0 \in \mathbb{R}\backslash C^{-1}(\mathcal{C}^\sigma)$. According to (52), $X \in \ker(\nabla_{C(t_0)}^*)$, so that the slice theorem implies $C(t) = C(t_0) + (t - t_0)X \in \mathcal{S}_{C(t_0),\varepsilon}$ for t close to t_0. If t_0 was an accumulation point of $\mathbb{R}\backslash C^{-1}(\mathcal{C}^\sigma)$, there would exist $t_1 \neq t_0$ such that $C(t_1) \in \mathcal{S}_{C(t_0),\varepsilon} \cap \mathcal{C}^{\sigma'}$ for some $\sigma' > \sigma$. By the properties of the slice, $\mathcal{G}_{C(t_1)} \subseteq \mathcal{G}_{C(t_0)}$. Since $C(t_1) = C(t_0) + (t_1 - t_0)X$, then $\mathcal{G}_X \supseteq \mathcal{G}_{C(t_1)}$. Writing $A = C(t_1) - t_1 X$ one sees that then $\mathcal{G}_A \subseteq \mathcal{G}_{C(t_1)}$ (contradiction). Hence, $\mathbb{R}\backslash C^{-1}(\mathcal{C}^\sigma)$ consists of isolated points. Due to closedness, it is then discrete.

6. Classification of gauge orbit types for $G = SU(n)$

Until now, complete classification results for the set of orbit types are known only for gauge group SU(n) and base manifolds of dimension up to four [65, 66], see also [67] for the discussion of a coarser stratification. In the following two sections these results will be reviewed. According to the reduction theorem, to determine the set of orbit types one has to work out the following programme:

1. classification of Howe subgroups of SU(n) up to conjugacy,
2. classification of Howe subbundles of P up to isomorphy,
3. specification of Howe subbundles which are holonomy-induced,
4. factorization by SU(n)-action,
5. determination of the natural partial ordering of Howe subbundles.

6.1. Howe subgroups of SU(n)

General references for the determination of Howe subgroups of classical Lie groups are [62, 68], see also [64] for the case of complex semisimple Lie algebras. For SU(n), however, it is not necessary to apply the general theory, because one can show, using the double commutant theorem, that the Howe subgroups of SU(n) are in 1–1 correspondence to unital $*$-subalgebras of $\mathrm{M}_n(\mathbb{C})$, the algebra of complex $n \times n$ matrices. The relation is given by intersecting the subalgebras with SU(n).

The unital $*$-subalgebras of $\mathrm{M}_n(\mathbb{C})$ can be described as follows. Let $\mathrm{K}(n)$ denote the set of pairs $J = (\mathbf{k}, \mathbf{m})$ of sequences $\mathbf{k} = (k_1, \ldots, k_r)$, $\mathbf{m} = (m_1, \ldots, m_r)$, $r = 1, \ldots, n$, consisting of positive integers such that

$$\mathbf{k} \cdot \mathbf{m} = \sum_{i=1}^{r} k_i m_i = n. \tag{54}$$

Any $J \in \mathrm{K}(n)$ defines a decomposition

$$\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i} \tag{55}$$

and an embedding

$$\prod_{i=1}^{r} \mathrm{M}_{k_i}(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) \qquad (D_1, \ldots, D_r) \mapsto \bigoplus_{i=1}^{r} D_i \otimes 1_{m_i}. \tag{56}$$

We denote the image of this embedding by $\mathrm{M}_J(\mathbb{C})$, its intersection with U(n) by U(J) and its intersection with SU(n) by SU(J). By construction, $\mathrm{M}_J(\mathbb{C})$ is a unital $*$-subalgebra of $\mathrm{M}_n(\mathbb{C})$. Conversely, it is not hard to show that any unital $*$-subalgebra of $\mathrm{M}_n(\mathbb{C})$ is conjugate to $\mathrm{M}_J(\mathbb{C})$ for some $J \in \mathrm{K}(n)$. Hence, up to conjugacy, the Howe subgroups of SU(n) are given by the subgroups SU(J), $J \in \mathrm{K}(n)$. Finally, it is evident that SU(J) and SU(J') are conjugate iff J' can be obtained from J by a simultaneous permutation of $\mathbf{k}$ and $\mathbf{m}$.

Remark. U(J) is the image of the restriction of (56) to $\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r)$. If we identify $\mathbb{C}^{k_i} \otimes \mathbb{C}^{m_i} \cong \mathbb{C}^{k_i m_i}$, $\left(c_1, \ldots, c_{k_i}\right) \otimes \left(d_1, \ldots, d_{m_i}\right) \mapsto \left(c_1 d_1, \ldots, c_{k_i} d_1, \ldots, c_1 d_{m_i}, \ldots, c_{k_i} d_{m_i}\right)$, the elements of U$(J)$ are given by matrices

$$\begin{pmatrix} \widetilde{D}_1 & 0 & \cdots & 0 \\ 0 & \widetilde{D}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{D}_r \end{pmatrix} \qquad \widetilde{D}_i = \begin{pmatrix} D_i & 0 & \cdots & 0 \\ 0 & D_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_i \end{pmatrix}$$

where $D_i \in \mathrm{U}(k_i)$ and $\widetilde{D}_i$ has dimension m_i. Then $\mathrm{SU}(J)$ consists of all such matrices which have determinant 1.

For later purposes, we introduce the following notation:

$$\begin{array}{rcccll} j_J & : & \mathrm{SU}(J) & \longrightarrow & \mathrm{U}(J) & \text{(embedding)} \\ i_J & : & \mathrm{U}(J) & \longrightarrow & \mathrm{U}(n) & \text{(embedding)} \\ \mathrm{pr}^{\mathrm{M}}_{J,i} & : & \mathrm{M}_J(\mathbb{C}) & \longrightarrow & \mathrm{M}_{k_i}(\mathbb{C}) & \text{(projection onto the } i\text{th factor)} \\ \mathrm{pr}^{\mathrm{U}}_{J,i} & : & \mathrm{U}(J) & \longrightarrow & \mathrm{U}(k_i) & \text{(projection onto the } i\text{th factor).} \end{array}$$

Let g denote the greatest common divisor of $\mathbf{m}$ and let $\widetilde{\mathbf{m}} = (\tilde{m}_1, \ldots, \tilde{m}_r)$ be defined by $m_i = g\tilde{m}_i, \forall i$. For any $D \in \mathrm{U}(J)$,

$$\det_{\mathrm{U}(n)}(D) = \prod_{i=1}^{r} \left[\det_{\mathrm{U}(k_i)}\left(\mathrm{pr}^{\mathrm{U}}_{J,i}(D)\right)\right]^{m_i}.$$

We can extract the gth root of the determinant by defining the Lie group homomorphism

$$\lambda^{\mathrm{U}}_J : \mathrm{U}(J) \longrightarrow \mathrm{U}(1) \qquad D \mapsto \prod_{i=1}^{r} \left[\det_{\mathrm{U}(k_i)}\left(\mathrm{pr}^{\mathrm{U}}_{J,i}(D)\right)\right]^{\tilde{m}_i}.$$

Then

$$\det_{\mathrm{U}(n)}(D) = \left[\lambda^{\mathrm{U}}_J(D)\right]^g \qquad \forall D \in \mathrm{U}(J).$$

Since $\lambda^{\mathrm{U}}_J(\mathrm{SU}(J)) = \mathbb{Z}_g \subseteq \mathrm{U}(1)$, λ^{U}_J induces a homomorphism $\lambda^{\mathrm{S}}_J : \mathrm{SU}(J) \to \mathbb{Z}_g$. We have the commutative diagram

$$\begin{tikzcd} \mathrm{SU}(J) \arrow[r, "j_J"] \arrow[d, "\lambda^S_J"'] & \mathrm{U}(J) \arrow[d, "\lambda^U_J"] \\ \mathbb{Z}_g \arrow[r, "j_g"'] & \mathrm{U}(1) \end{tikzcd} \tag{57}$$

where j_g denotes natural embedding.

Below we shall need the low dimensional homotopy groups of $\mathrm{SU}(J)$. In dimension $k \geqslant 1$, they can be derived in a standard way from the corresponding homotopy groups of $\mathrm{U}(J) \cong \mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r)$ by means of the exact homotopy sequence of the $\mathrm{SU}(J)$-bundle $\det_{\mathrm{U}(n)} : \mathrm{U}(J) \to \mathrm{U}(1)$. In dimension $k = 0$ we have, by definition, $\pi_0(\mathrm{SU}(J)) = \mathrm{SU}(J)/\mathrm{SU}(J)_0$, where $\mathrm{SU}(J)_0$ denotes the connected component of the identity of $\mathrm{SU}(J)$. One can show $\mathrm{SU}(J)/\mathrm{SU}(J)_0 \cong \mathbb{Z}_g$, with the isomorphism being induced by λ^{S}_J, see [65, lemma 5.2]. Thus

$$\pi_k(\mathrm{SU}(J)) = \begin{cases} \mathbb{Z}_g & | \quad k = 0 \\ \mathbb{Z}^{\oplus(r-1)} & | \quad k = 1 \\ \pi_k(\mathrm{U}(k_1)) \oplus \cdots \oplus \pi_k(\mathrm{U}(k_r)) & | \quad k > 1. \end{cases} \tag{58}$$

6.2. *Howe subbundles of SU(n)-bundles*

In this subsection, let $J \in \mathrm{K}(n)$ be arbitrary but fixed. We are going to derive a classification, up to isomorphy, of principal $\mathrm{SU}(J)$-bundles over M in terms of appropriately chosen characteristic classes. Recall that we assume $\dim(M) \leqslant 4$. Then, on the level of these classes, we shall obtain a characterization of those $\mathrm{SU}(J)$-bundles which are reductions of a given $\mathrm{SU}(n)$-bundle P. In the following we use some facts from bundle theory as well as from algebraic topology. For a brief account, see appendices A and B.

Generally, each isomorphism class of principal $\mathrm{SU}(J)$-bundles over M is in 1–1 correspondence to a homotopy class of maps from M to the classifying space $\mathrm{BSU}(J)$ of $\mathrm{SU}(J)$, its so-called classifying map. As usual, we denote the set of all homotopy classes by $[M, \mathrm{BSU}(J)]$. Due to the potentially complicated structure of the space $\mathrm{BSU}(J)$, $[M, \mathrm{BSU}(J)]$ is hardly tractable in full generality. However, we can use three major inputs from algebraic topology to get control of it under our specific assumption $\dim(M) \leqslant 4$.

First, assume that we are able to find a simpler space $\mathrm{BSU}(J)_n$ and a map $f_n : \mathrm{BSU}(J) \to \mathrm{BSU}(J)_n$ such that the homomorphism induced by f_n on homotopy groups is an isomorphism in dimension $k < n$ and surjective in dimension n. Then composition with f_n defines a bijection from $[M, \mathrm{BSU}(J)]$ onto $[M, \mathrm{BSU}(J)_n]$, see [20, chapter VII]. We remark that $\mathrm{BSU}(J)_n$ is called an n-equivalent approximation of $\mathrm{BSU}(J)$ and f_n is called an n-equivalence.

Second, algebraic topology provides a method to successively construct n-equivalent approximations, starting from $n = 1$: the method of Postnikov tower. It renders $\mathrm{BSU}(J)_n$ as an n-stage fibration over a point, where the fibre at stage k is given by the Eilenberg–MacLane space $\mathrm{K}(\pi_k(\mathrm{BSU}(J)), k)$. This space is defined as a CW complex, up to homotopy equivalence, by the property that its only nonvanishing homotopy group is $\pi_k(\mathrm{BSU}(J))$ in dimension k. Recall that $\pi_k(\mathrm{BSU}(J)) \cong \pi_{k-1}(\mathrm{SU}(J))$. For the precise formulation of the method see appendix B. For a detailed explanation as well as an application to standard groups, we refer to [9].

Applying the method of the Postnikov tower to $\mathrm{BSU}(J)$ up to stage 5 we obtain, see [65, theorem 5.4],

$$\mathrm{BSU}(J)_5 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2) \times \prod_{j=1}^{r^*} K(\mathbb{Z}, 4) \tag{59}$$

where r^* denotes the number of members $k_i > 1$. For the convenience of the reader we give the proof of (59) in appendix C. We note that the successive fibrations mentioned above turn out to be trivial here, i.e. they are just direct products. As a consequence, we have a bijection

$$\begin{aligned} [M, \mathrm{BSU}(J)] &\to [M, K(\mathbb{Z}_g, 1)] \times \prod_{i=1}^{r-1}[M, K(\mathbb{Z}, 2)] \times \prod_{i=1}^{r^*}[M, K(\mathbb{Z}, 4)] \\ f &\mapsto \left(\mathrm{pr}_1 \circ f_5 \circ f, \{\mathrm{pr}_{2i} \circ f_5 \circ f\}_{i=1}^{r-1}, \{\mathrm{pr}_{4i} \circ f_5 \circ f\}_{i=1}^{r^*}\right), \end{aligned}$$

where $f_5 : \mathrm{BSU}(J) \to \mathrm{BSU}(J)_5$ is a 5-equivalence and the pr_{ij} are the projections from $\mathrm{BSU}(J)_5$ onto its factors.

To treat the factors on the rhs we use a third input from algebraic topology. We will explain it for $[M, K(\mathbb{Z}_g, 1)]$. Namely, the theory of Eilenberg–MacLane spaces provides the following relation between homotopy and cohomology, see appendix B. There exists $\gamma_1 \in H^1(\mathrm{K}(\mathbb{Z}_g, 1), \mathbb{Z}_g)$ (the first $\mathbb{Z}_g$-valued cohomology group) such that the assignment

$$[M, K(\mathbb{Z}_g, 1)] \to H^1(M, \mathbb{Z}_g) \qquad \mathrm{pr}_1 \circ f_5 \circ f \mapsto (\mathrm{pr}_1 \circ f_5 \circ f)^* \gamma_1 \tag{61}$$

is a bijection. Here $(\mathrm{pr}_1 \circ f_5 \circ f)^*$ denotes the homomorphisms induced in cohomology. Writing $(\mathrm{pr}_1 \circ f_5 \circ f)^* \gamma_1 = f^*(\mathrm{pr}_1 \circ f_5)^* \gamma_1$, we observe that the bijection (61) is characterized by the image under f^* of the fixed element $(\mathrm{pr}_1 \circ f_5)^* \gamma_1$ of $H^1(\mathrm{BSU}(J), \mathbb{Z}_g)$. Thus, if for given maps $f, f' : M \to \mathrm{BSU}(J)$ the induced homomorphisms $f^*, f'^* : H^1(\mathrm{BSU}(J), \mathbb{Z}_g) \to H^1(M, \mathbb{Z}_g)$ coincide then the maps $\mathrm{pr}_1 \circ f_5 \circ f$ and $\mathrm{pr}_1 \circ f_5 \circ f'$ are homotopic. Analogously, one finds for $k = 2, 4$ that if the induced homomorphisms $f^*, f'^* : H^k(\mathrm{BSU}(J), \mathbb{Z}) \to H^k(M, \mathbb{Z})$ coincide then $\mathrm{pr}_{ki} \circ f_5 \circ f$ and $\mathrm{pr}_{ki} \circ f_5 \circ f'$ are homotopic, for all admissible i. Thus, considering that (60) is a bijection, we arrive at the following result: Two maps $f, f' : M \to \mathrm{BSU}(J)$ are homotopic if they induce the same homomorphisms on the cohomology groups $H^1(\mathrm{BSU}(J), \mathbb{Z}_g)$, $H^2(\mathrm{BSU}(J), \mathbb{Z})$ and $H^4(\mathrm{BSU}(J), \mathbb{Z})$. Thus, to characterize homotopy classes of maps $M \to \mathrm{BSU}(J)$, as usual, we have to determine a set of generators for these cohomology groups and to evaluate f^* on them. In this way, a set of characteristic classes is associated with any element of $[M, \mathrm{BSU}(J)]$, hence to any $\mathrm{SU}(J)$-bundle through its classifying map. This set is complete in the sense that coincidence of characteristic classes implies isomorphy of the corresponding bundles.

To construct a set of generators, we use the commutative diagram (57), which on the level of classifying spaces reads

$$\begin{array}{ccc} \mathrm{BSU}(J) & \xrightarrow{B_{j_J}} & \mathrm{BU}(J) \\ {\scriptstyle \mathrm{B}\lambda^{\mathrm{S}}_J}\downarrow & & \downarrow{\scriptstyle \mathrm{B}\lambda^{\mathrm{U}}_J}. \\ \mathrm{B}\mathbb{Z}_g & \xrightarrow[\mathrm{B}j_g]{} & \mathrm{BU}(1) \end{array} \tag{62}$$

First, consider the $\mathbb{Z}$-valued cohomology. Recall that the cohomology algebra $H^*(\mathrm{BU}(k_i), \mathbb{Z})$ is generated freely over $\mathbb{Z}$ by elements $\gamma^{(2j)}_{\mathrm{U}(k_i)} \in H^{2j}(\mathrm{BU}(k_i), \mathbb{Z})$, $j = 1, \ldots, k_i$, see [13]. We denote

$$\gamma_{\mathrm{U}(k_i)} = 1 + \gamma^{(2)}_{\mathrm{U}(k_i)} + \cdots + \gamma^{(2k_i)}_{\mathrm{U}(k_i)}. \tag{63}$$

The generators $\gamma^{(2j)}_{\mathrm{U}(k_i)}$ define elements

$$\tilde{\gamma}^{(2j)}_{J,i} = \left(\mathrm{Bpr}^{\mathrm{U}}_{J,i}\right)^* \gamma^{(2j)}_{\mathrm{U}(k_i)} \tag{64}$$

$$\gamma^{(2j)}_{J,i} = (\mathrm{B}j_J)^* \, \tilde{\gamma}^{(2j)}_{J,i} \tag{65}$$

of $H^{2j}(\mathrm{BU}(J), \mathbb{Z})$ and $H^{2j}(\mathrm{BSU}(J), \mathbb{Z})$, respectively. We denote

$$\tilde{\gamma}_{J,i} = 1 + \tilde{\gamma}^{(2)}_{J,i} + \cdots + \tilde{\gamma}^{(2k_i)}_{J,i} \qquad \tilde{\gamma}_J = (\tilde{\gamma}_{J,1}, \ldots, \tilde{\gamma}_{J,r}) \tag{66}$$

$$\gamma_{J,i} = 1 + \gamma^{(2)}_{J,i} + \cdots + \gamma^{(2k_i)}_{J,i} \qquad \gamma_J = (\gamma_{J,1}, \ldots, \gamma_{J,r}). \tag{67}$$

It is a direct consequence of the Künneth theorem for cohomology that the cohomology algebra $H^*(\mathrm{BU}(J), \mathbb{Z})$ is freely generated over $\mathbb{Z}$ by the elements $\tilde{\gamma}^{(2j)}_{J,i}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, r$. Moreover, considering that $\mathrm{B}j_J : \mathrm{BSU}(J) \to \mathrm{BU}(J)$ is a $\mathrm{U}(1)$-bundle and, therefore, induces a Gysin sequence one can show that $(\mathrm{B}j_J)^*$ is surjective, see [65, lemma 5.7]. Thus, the cohomology algebra $H^*(\mathrm{BSU}(J), \mathbb{Z})$ is generated over $\mathbb{Z}$ by the elements $\gamma^{(2j)}_{J,i}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, r$. We remark that the generators $\gamma^{(2)}_{J,i}$ of $H^*(\mathrm{BSU}(J), \mathbb{Z})$ are subject to a relation, which is however irrelevant for our purposes, because it follows from another relation to be derived below.

Next, we have to consider $H^1(\mathrm{BSU}(J), \mathbb{Z}_g)$. We note the following facts:

(i) The induced homomorphism $\left(\mathrm{B}\lambda^{\mathrm{S}}_J\right)^* : H^1(\mathrm{B}\mathbb{Z}_g, \mathbb{Z}_g) \to H^1(\mathrm{BSU}(J), \mathbb{Z}_g)$ is an isomorphism. This follows by virtue of the Hurewicz and universal coefficient theorems from the obvious fact that λ^{S}_J induces an isomorphism of homotopy groups $\pi_0(\mathrm{SU}(J)) \to \pi_0(\mathbb{Z}_g)$.

(ii) From the (long) exact sequence induced by the short exact sequence of coefficient groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_g \to 0$ one can read off that the associated Bockstein homomorphism $\beta_g : H^1(\mathrm{B}\mathbb{Z}_g, \mathbb{Z}_g) \to H^2(\mathrm{B}\mathbb{Z}_g, \mathbb{Z})$ is an isomorphism.

(iii) The surjectivity of $\mathrm{B}j_J$, mentioned above, implies, in particular, surjectivity of the homomorphism $(\mathrm{B}j_g)^* : H^2(\mathrm{BU}(1), \mathbb{Z}) \to H^2(\mathrm{B}\mathbb{Z}_g, \mathbb{Z})$.

It follows that $H^1(\mathrm{BSU}(J), \mathbb{Z}_g)$ is generated by the single element

$$\delta_J := \left(\mathrm{B}\lambda_J^{\mathrm{S}}\right)^* \beta_g^{-1} (\mathrm{B}j_g)^* \gamma_{\mathrm{U}(1)}^{(2)}. \tag{68}$$

Finally, the commutative diagram (62) induces a relation between the generators $\gamma_{J,i}^{(2)}$ and δ_J. To formulate it, we introduce the following notation. For any topological space X and any sequence of nonnegative integers $\mathbf{b} = (b_1, \ldots, b_s)$, define a polynomial function

$$E_{\mathbf{b}} : \prod_{i=1}^{s} H^{\mathrm{even}}(X, \mathbb{Z}) \to H^{\mathrm{even}}(X, \mathbb{Z}) \qquad (\alpha_1, \ldots, \alpha_s) \mapsto \alpha_1^{b_1} \ldots \alpha_s^{b_s}. \tag{69}$$

One can check the following formulae for the components of $E_{\mathbf{b}}$ in degree 2 and 4:

$$E_{\mathbf{b}}^{(2)}(\alpha) = \sum_{i=1}^{s} b_i \alpha_i^{(2)} \tag{70}$$

$$E_{\mathbf{b}}^{(4)}(\alpha) = \sum_{i=1}^{s} b_i \alpha_i^{(4)} + \sum_{i=1}^{s} \frac{b_i(b_i-1)}{2} \alpha_i^{(2)} \alpha_i^{(2)} + \sum_{i<j=2}^{s} b_i b_j \alpha_i^{(2)} \alpha_j^{(2)}. \tag{71}$$

A straightforward computation, see [65, lemma 5.12], yields

$$\left(\mathrm{B}\lambda_J^{\mathrm{U}}\right)^* \gamma_{\mathrm{U}(1)}^{(2)} = E_{\tilde{\mathbf{m}}}^{(2)}(\tilde{\gamma}_J). \tag{72}$$

Then the commutative diagram (62) implies

$$E_{\tilde{\mathbf{m}}}^{(2)}(\gamma_J) = (\mathrm{B}j_J)^* E_{\tilde{\mathbf{m}}}^{(2)}(\tilde{\gamma}_J) = (\mathrm{B}j_J)^* \left(\mathrm{B}\lambda_J^{\mathrm{U}}\right)^* \gamma_{\mathrm{U}(1)}^{(2)} = \left(\mathrm{B}\lambda_J^{\mathrm{S}}\right)^* (\mathrm{B}j_g)^* \gamma_{\mathrm{U}(1)}^{(2)}.$$

Thus, by definition of δ_J, the relation is

$$E_{\tilde{\mathbf{m}}}^{(2)}(\gamma_J) = \beta_g(\delta_J). \tag{73}$$

The generators $\gamma_{J,i}^{(2j)}$, δ_J constructed above define the following characteristic classes for $\mathrm{SU}(J)$-bundles Q over M:

$$\xi_J(Q) := (f_Q)^* \delta_J$$

$$\alpha_{J,i}^{(2j)}(Q) := (f_Q)^* \gamma_{J,i}^{(2j)} \qquad j = 1, \ldots, k_i \quad i = 1, \ldots, r.$$

Here f_Q denotes the classifying map of Q. We denote $\alpha_{J,i} = 1 + \alpha_{J,i}^{(2)} + \cdots + \alpha_{J,i}^{(2k_i)}$ and $\alpha_J = (\alpha_{J,1}, \ldots, \alpha_{J,r})$. Due to (73), α_J and ξ_J are subject to the relation

$$E_{\tilde{\mathbf{m}}}^{(2)}(\alpha_J(Q)) = \beta_g(\xi_J(Q)). \tag{74}$$

By construction, the characteristic classes so defined have the following interpretation in terms of ordinary characteristic classes of certain bundles naturally associated with Q. First, by extending the structure group of Q to $\mathrm{U}(J)$ we obtain a $\mathrm{U}(J)$-bundle $\tilde{Q}$. Since $\mathrm{U}(J) \cong \mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r)$, $\tilde{Q}$ decomposes into a Whitney product of $\mathrm{U}(k_i)$-bundles $\tilde{Q}_i$. Formally, $\tilde{Q}_i$ is given by the associated bundle $Q \times_{\mathrm{SU}(J)} \mathrm{U}(k_i)$, where $\mathrm{SU}(J)$ acts via $\mathrm{pr}_{J,i}^{\mathrm{U}} \circ j_J$ by left multiplication on $\mathrm{U}(k_i)$. Hence, its classifying map is $\mathrm{Bpr}_{J,i}^{\mathrm{U}} \circ \mathrm{B}j_J \circ f_Q$, see formula (A.3) in appendix A. Using this, a standard calculation yields

$$\alpha_{J,i}(Q) = c(\tilde{Q}_i) \tag{75}$$

where c denotes the total Chern class. Second, factorizing Q by $\mathrm{SU}(J)_0$, the connected component of the identity of $\mathrm{SU}(J)$, we obtain a $\mathbb{Z}_g$-bundle Q_0. It is given by the associated bundle $Q \times_{\mathrm{SU}(J)} \mathbb{Z}_g$, where $\mathrm{SU}(J)$ acts on $\mathbb{Z}_g$ via the homomorphism λ_J^{S}. Then formula (A.3) implies that Q_0 has classifying map $\mathrm{B}\lambda_J^{\mathrm{S}} \circ f_Q$. This allows calculation of

$$\xi_J(Q) = \chi_g(Q_0) \tag{76}$$

where χ_g is a (suitably chosen) generating characteristic class for $\mathbb{Z}_g$-bundles over M.

We remark that the commutative diagram (57) implies that extension of Q_0 to structure group $\mathrm{U}(1)$ and factorization of $\tilde{Q}$ by $\mathrm{SU}(J)_0$ yield isomorphic $\mathrm{U}(1)$-bundles. In this way, the relation (74) expresses itself on the level of the associated bundles.

So far, we have found that the classes α_J and ξ_J assign to any $\mathrm{SU}(J)$-bundle Q over M an element of the set

$$\mathrm{K}(M, J) = \left\{ (\alpha, \xi) \in \prod_{i=1}^{r} \prod_{j=1}^{k_i} H^{2j}(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}_g) \,\middle|\, E^{(2)}_{\tilde{\mathbf{m}}}(\alpha) = \beta_g(\xi) \right\}.$$

We already know that $\alpha_J(Q) = \alpha_J(Q')$ and $\xi_J(Q) = \xi_J(Q')$ imply $Q \cong Q'$. Thus, for $\mathrm{K}(M, J)$ to be a classifying set for $\mathrm{SU}(J)$-bundles it remains to prove that for any of its elements a bundle with the corresponding characteristic classes exists. Thus, let (α, ξ) be given. There exist

(i) $\mathrm{U}(k_i)$-bundles $\tilde{Q}_i$ such that $c(\tilde{Q}_i) = \alpha_i$. Their Whitney product defines a $\mathrm{U}(J)$-bundle $\tilde{Q}$.
(ii) a $\mathbb{Z}_g$-bundle Q_0 such that $\chi_g(Q_0) = \xi$.

The defining relation of $\mathrm{K}(M, J)$ ensures that Q_0 is a reduction of the quotient bundle $\tilde{Q}/\mathrm{SU}(J)_0$, see [65, lemma 5.15]. Then the pre-image Q of Q_0 in $\tilde{Q}$ is an $\mathrm{SU}(J)$-bundle. By construction, (75) and (76) hold. Hence, we have $\alpha_J(Q) = \alpha$ and $\xi_J(Q) = \xi$.

We summarize.

Theorem 6.1. *Let M be a manifold,* $\dim M \leqslant 4$*, and let* $J \in \mathrm{K}(n)$*. Then the characteristic classes α_J and ξ_J define a bijection from isomorphism classes of principal* $\mathrm{SU}(J)$*-bundles over M onto* $\mathrm{K}(M, J)$*.*

Next, we have to characterize the $\mathrm{SU}(J)$-bundles Q that are reductions of a given $\mathrm{SU}(n)$-bundle P. Evidently, $Q \subseteq P$ iff P can be obtained from Q by extending the structure group to $\mathrm{SU}(n)$, or iff the extensions of P and $\tilde{Q}$ to structure group $\mathrm{U}(n)$ coincide. A standard calculation yields that the total Chern class of the extension of $\tilde{Q}$ is given by $E_{\mathbf{m}}(\alpha_J(Q))$. Thus, using the notation

$$\mathrm{K}(P, J) = \{(\alpha, \xi) \in \mathrm{K}(M, J) \mid E_{\mathbf{m}}(\alpha) = c(P)\}$$

we have

Theorem 6.2. *Let P be a principal* $\mathrm{SU}(n)$*-bundle over a manifold M,* $\dim M \leqslant 4$*, and let* $J \in \mathrm{K}(n)$*. Then the characteristic classes α_J, ξ_J define a bijection from isomorphism classes of reductions of P to the subgroup* $\mathrm{SU}(J)$ *onto* $\mathrm{K}(P, J)$*.*

The equation $E_{\mathbf{m}}(\alpha) = c(P)$ actually contains the two equations $E^{(2)}_{\mathbf{m}}(\alpha) = 0$ and $E^{(4)}_{\mathbf{m}}(\alpha) = c_2(P)$. However, under the assumption $(\alpha, \xi) \in \mathrm{K}(M, J)$, the first one is redundant, because due to (70), $E^{(2)}_{\mathbf{m}}(\alpha) = g E^{(2)}_{\tilde{\mathbf{m}}}(\alpha) = g\beta_g(\xi) = 0$. Thus, the relevant equations are

$$E^{(2)}_{\tilde{\mathbf{m}}}(\alpha) = \beta_g(\xi) \tag{77}$$

$$E^{(4)}_{\mathbf{m}}(\alpha) = c_2(P). \tag{78}$$

The set of solutions of (77) yields $\mathrm{K}(M, J)$, the set of solutions of both equations (77) and (78) yields $\mathrm{K}(P, J)$.

This concludes the classification of Howe subbundles of P, i.e. step 2 of our programme. We have found that, up to the principal action of $\mathrm{SU}(n)$, the Howe subbundles are given by triples $(J;\alpha,\xi)$, where $J\in \mathrm{K}(n)$ and $(\alpha,\xi)\in \mathrm{K}(P,J)$. For further use, let us denote the set of all such triples by $\mathrm{K}(P)$. It may be viewed as the disjoint union of all $\mathrm{K}(P,J)$, $J\in\mathrm{K}(n)$. Moreover, for given $L\in\mathrm{K}(P)$, $L=(J;\alpha,\xi)$, let Q_L denote the corresponding Howe subbundle. That is, Q_L is the reduction of P to $\mathrm{SU}(J)$ which has characteristic classes $\alpha_J(Q_L)=\alpha$ and $\xi_J(Q_L)=\xi$. It is unique up to isomorphy.

6.3. Examples

We determine $\mathrm{K}(P,J)$ for several specific values of J and for base manifolds $M=\mathrm{S}^4,\mathrm{S}^2\times\mathrm{S}^2,\mathrm{T}^4$, and $\mathrm{L}^3_p\times\mathrm{S}^1$. Here L^3_p denotes the three-dimensional lens space which is defined to be the quotient of the restriction of the natural action of U(1) on the sphere $\mathrm{S}^3\subset\mathbb{C}^2$ to the subgroup $\mathbb{Z}_p$. Note that L^3_p is orientable.

Let us derive the respective Bockstein homomorphisms $\beta_g : H^1(M,\mathbb{Z}_g)\to H^2(M,\mathbb{Z})$. Since the Abelian group $H^1(M,\mathbb{Z}_g)$ has vanishing free part and since for products of spheres the integer-valued second cohomology is free Abelian, the Bockstein homomorphism is trivial here. For $M=\mathrm{L}^3_p\times\mathrm{S}^1$, on the other hand, let $\gamma^{(1)}_{\mathrm{L}^3_p;\mathbb{Z}_g}$ and $\gamma^{(1)}_{\mathrm{S}^1}$ be generators of $H^1\big(\mathrm{L}^3_p,\mathbb{Z}_g\big)$ and $H^1(\mathrm{S}^1,\mathbb{Z})$, respectively. One has $H^1\big(\mathrm{L}^3_p\times\mathrm{S}^1,\mathbb{Z}_g\big)=\mathbb{Z}_{\langle p,g\rangle}\oplus\mathbb{Z}_g$, where $\langle p,g\rangle$ denotes the greatest common divisor of p and g. Here the first factor is generated by $\gamma^{(1)}_{\mathrm{L}^3_p;\mathbb{Z}_g}\times 1_{\mathrm{S}^1}$ and the second one by $1_{\mathrm{L}^3_p;\mathbb{Z}_g}\times\gamma^{(1)}_{\mathrm{S}^1}$. In terms of these generators and an appropriately chosen generator $\gamma^{(2)}_{\mathrm{L}^3_p;\mathbb{Z}}$ of $H^2\big(\mathrm{L}^3_p,\mathbb{Z}\big)\cong\mathbb{Z}_p$, the Bockstein homomorphism is

$$\beta_g\left(\gamma^{(1)}_{\mathrm{L}^3_p;\mathbb{Z}_g}\times 1_{\mathrm{S}^1}\right)=\frac{p}{\langle p,g\rangle}\gamma^{(2)}_{\mathrm{L}^3_p;\mathbb{Z}}\times 1_{\mathrm{S}^1}\qquad \beta_g\left(1_{\mathrm{L}^3_p;\mathbb{Z}_g}\times\gamma^{(1)}_{\mathrm{S}^1}\right)=0. \tag{79}$$

Now we discuss specific J. We write them in the form $J=(k_1,\ldots,k_r|m_1,\ldots,m_r)$.

Example 1. $J=(1|n)\in\mathrm{K}(n)$. Here $\mathrm{SU}(J)=\mathbb{Z}_n$, the centre of $\mathrm{SU}(n)$. Moreover, $g=n$. The variables are $\xi\in H^1(M,\mathbb{Z}_n)$ and $\alpha=1+\alpha^{(2)},\alpha^{(2)}\in H^2(M,\mathbb{Z})$. The system of equations (77) and (78) reads

$$\alpha^{(2)}=\beta_n(\xi) \tag{80}$$

$$\frac{n(n-1)}{2}(\alpha^{(2)})^2=c_2(P). \tag{81}$$

Equation (80) yields $n\alpha^{(2)}=0$, so that equation (81) requires $c_2(P)=0$. Thus, $\mathrm{K}(P,J)$ is nonempty iff P is trivial and is then parametrized by ξ. This coincides with what is known about $\mathbb{Z}_n$-reductions of $\mathrm{SU}(n)$-bundles.

Example 2. $J=(n|1)\in\mathrm{K}(n)$. Here $\mathrm{SU}(J)=\mathrm{SU}(n)$, the whole group. Due to $g=1$, the only variable is $\alpha=1+\alpha^{(2)}+\alpha^{(4)}$. Equations (77) and (78) read $\alpha^{(2)}=0$ and $\alpha^{(4)}=c_2(P)$, respectively. Thus, $\mathrm{K}(P,J)$ consists of P itself.

Example 3. $J=(1,1|2,2)\in\mathrm{K}(4)$. Here $g=2$. One can check that $\mathrm{SU}(J)$ has connected components $\{\mathrm{diag}(z,z,z^{-1},z^{-1})|z\in\mathrm{U}(1)\}$ and $\{\mathrm{diag}(z,z,-z^{-1},-z^{-1})|z\in\mathrm{U}(1)\}$. It is therefore isomorphic to $\mathrm{U}(1)\times\mathbb{Z}_2$. The variables are $\xi\in H^1(M,\mathbb{Z}_2)$ and $\alpha_i=1+\alpha^{(2)}_i$, $i=1,2$. The system of equations under consideration is

$$\alpha^{(2)}_1+\alpha^{(2)}_2=\beta_2(\xi) \tag{82}$$

$$\left(\alpha^{(2)}_1\right)^2+\left(\alpha^{(2)}_2\right)^2+4\alpha^{(2)}_1\alpha^{(2)}_2=c_2(P). \tag{83}$$

We solve equation (82) w.r.t. $\alpha_2^{(2)}$ and insert it into equation (83). Since, due to compactness and orientability of M, $H^4(M,\mathbb{Z})$ is torsion-free, products including $\beta_2(\xi)$ vanish. Thus, we obtain that ξ can be chosen arbitrarily, whereas $\alpha_1^{(2)}$ must solve the equation

$$-2\left(\alpha_1^{(2)}\right)^2 = c_2(P). \tag{84}$$

Let us discuss the result for the different base manifolds.

(i) $M = \mathrm{S}^4$: due to $H^1(M,\mathbb{Z}_2) = 0$ and $H^2(M,\mathbb{Z}) = 0$, $\mathrm{K}(P,J)$ is nonempty iff $c_2(P) = 0$, in which case it contains the (necessarily trivial) $\mathrm{U}(1)\times\mathbb{Z}_2$-bundle over S^4.

(ii) $M = \mathrm{L}_p^3\times\mathrm{S}^1$: we have $H^1(M,\mathbb{Z}_2)\cong\mathbb{Z}_{\langle 2,p\rangle}\oplus\mathbb{Z}_2$ and $H^2(M,\mathbb{Z})\cong\mathbb{Z}_p$. In particular, $\left(\alpha_1^{(2)}\right)^2 = 0$. Hence, if $c_2(P) = 0$, $\mathrm{K}(P,J) = \left(\mathbb{Z}_{\langle 2,p\rangle}\oplus\mathbb{Z}_2\right)\times\mathbb{Z}_p$. Otherwise, $\mathrm{K}(P,J) = \emptyset$.

(iii) $M = \mathrm{S}^2\times\mathrm{S}^2$: we have $H^1(M,\mathbb{Z}_2) = 0$ and $H^2(M,\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}$. The latter is generated by $\gamma_{\mathrm{S}^2}^{(2)}\times 1_{\mathrm{S}^2}$ and $1_{\mathrm{S}^2}\times\gamma_{\mathrm{S}^2}^{(2)}$, where $\gamma_{\mathrm{S}^2}^{(2)}$ is a generator of $H^2(\mathrm{S}^2,\mathbb{Z})$. Then $H^4(M,\mathbb{Z})$ is generated by $\gamma_{\mathrm{S}^2}^{(2)}\times\gamma_{\mathrm{S}^2}^{(2)}$. Writing

$$\alpha_1^{(2)} = a\gamma_{\mathrm{S}^2}^{(2)}\times 1_{\mathrm{S}^2} + b\,1_{\mathrm{S}^2}\times\gamma_{\mathrm{S}^2}^{(2)} \tag{85}$$

with $a, b\in\mathbb{Z}$, equation (84) becomes

$$-4ab\gamma_{\mathrm{S}^2}^{(2)}\times\gamma_{\mathrm{S}^2}^{(2)} = c_2(P). \tag{86}$$

If $c_2(P) = 0$, there are two series of solutions: $a = 0$ and $b\in\mathbb{Z}$ as well as $a\in\mathbb{Z}$ and $b = 0$. Here $\mathrm{K}(P,J)$ is infinite. If $c_2(P) = 4l\gamma_{\mathrm{S}^2}^{(2)}\times\gamma_{\mathrm{S}^2}^{(2)}$, $l\neq 0$, then $a = q$ and $b = -l/q$, where q runs through the (positive and negative) divisors of l. Hence, in this case, the cardinality of $\mathrm{K}(P,J)$ is twice the number of divisors of l. If $c_2(P)$ is not divisible by 4 then $\mathrm{K}(P,J) = \emptyset$.

(iv) $M = \mathrm{T}^4$: here $H^1(M,\mathbb{Z}_2)\cong\mathbb{Z}_2^{\oplus 4}$ and $H^2(M,\mathbb{Z})\cong\mathbb{Z}^{\oplus 6}$. The latter is generated by elements $\gamma_{\mathrm{T}^4;ij}^{(2)}$, $1\leqslant i<j\leqslant 4$, where $\gamma_{\mathrm{T}^4;12}^{(2)} = \gamma_{\mathrm{S}^1}^{(1)}\times\gamma_{\mathrm{S}^1}^{(1)}\times 1_{\mathrm{S}^1}\times 1_{\mathrm{S}^1}$, $\gamma_{\mathrm{T}^4;13}^{(2)} = \gamma_{\mathrm{S}^1}^{(1)}\times 1_{\mathrm{S}^1}\times\gamma_{\mathrm{S}^1}^{(1)}\times 1_{\mathrm{S}^1}$ etc. $H^4(M,\mathbb{Z})$ is generated by $\gamma_{\mathrm{T}^4}^{(4)} = \gamma_{\mathrm{S}^1}^{(1)}\times\gamma_{\mathrm{S}^1}^{(1)}\times\gamma_{\mathrm{S}^1}^{(1)}\times\gamma_{\mathrm{S}^1}^{(1)}$. One can check $\gamma_{\mathrm{T}^4;ij}^{(2)}\gamma_{\mathrm{T}^4;kl}^{(2)} = \epsilon_{ijkl}\,\gamma_{\mathrm{T}^4}^{(4)}$, where ϵ_{ijkl} denotes the totally antisymmetric tensor in four dimensions. Writing $\alpha_1^{(2)} = \sum_{1\leqslant i<j\leqslant 4} a_{ij}\gamma_{\mathrm{T}^4;ij}^{(2)}$, equation (84) becomes

$$-4\left(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}\right)\gamma_{\mathrm{T}^4}^{(4)} = c_2(P).$$

Hence, again $\mathrm{K}(P,J)\neq\emptyset$ iff $c_2(P)$ is divisible by 4, in which case now it always has infinitely many elements.

Example 4. $J = (1,1\,|\,2,3)\in\mathrm{K}(5)$. The subgroup $\mathrm{SU}(J)$ of $\mathrm{SU}(5)$ consists of matrices of the form $\mathrm{diag}(z_1,z_1,z_2,z_2,z_2)$, where $z_1,z_2\in\mathrm{U}(1)$ such that $z_1^2\,z_2^3 = 1$. We can parametrize $z_1 = z^3$, $z_2 = z^{-2}$, $z\in\mathrm{U}(1)$. Hence, $\mathrm{SU}(J)$ is isomorphic to $\mathrm{U}(1)$. The variables are $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2$. The equations to be solved read

$$2\alpha_1^{(2)} + 3\alpha_2^{(2)} = 0 \tag{87}$$

$$\left(\alpha_1^{(2)}\right)^2 + 3\left(\alpha_2^{(2)}\right)^2 + 6\alpha_1^{(2)}\alpha_2^{(2)} = c_2(P). \tag{88}$$

Equation (87) can be parametrized by $\alpha_1^{(2)} = 3\eta$, $\alpha_2^{(2)} = -2\eta$, where $\eta\in H^2(M,\mathbb{Z})$. Then (88) becomes $-15\eta^2 = c_2(P)$. The discussion of this equation is analogous to that of equation (84) above. For example, in the case $M = \mathrm{S}^2\times\mathrm{S}^2$, $\mathrm{K}(P,J)\neq\emptyset$ iff $c_2(P)$ is divisible by 15.

Example 5. $J = (2, 3|1, 1) \in \mathrm{K}(5)$. Here $\mathrm{SU}(J) \cong \mathrm{S}[\mathrm{U}(2) \times \mathrm{U}(3)]$. This is the symmetry group of the standard model. In the grand unified SU(5)-model it is the subgroup to which SU(5) is broken by the heavy Higgs field. Moreover, it is the centralizer of the subgroup discussed in example 4.

Since $g = 1$, the variables are $\alpha_i = 1 + \alpha_i^{(2)} + \alpha_i^{(4)}$, $i = 1, 2$. Equations (77) and (78) read

$$\alpha_1^{(2)} + \alpha_2^{(2)} = 0 \tag{89}$$

$$\alpha_1^{(4)} + \alpha_2^{(4)} + \alpha_1^{(2)}\alpha_2^{(2)} = c_2(P). \tag{90}$$

Using (89) to replace $\alpha_2^{(2)}$ in (90) we obtain for the latter $\alpha_2^{(4)} = c_2(P) - \alpha_1^{(4)} + \left(\alpha_1^{(2)}\right)^2$. Thus, $\mathrm{K}(P, J)$ can be parametrized by α_1 (or α_2), i.e. by the Chern class of one of the factors U(2) or U(3). This is well known [50].

Example 6. $J = (2|2)$. We have $g = 2$. The subgroup $\mathrm{SU}(J)$ of SU(4) consists of matrices $D \oplus D$, where $D \in \mathrm{U}(2)$ such that $(\det D)^2 = 1$. Hence, it has connected components $\{D \oplus D | D \in \mathrm{SU}(2)\}$ and $\{(\mathrm{i}D) \oplus (\mathrm{i}D) | D \in \mathrm{SU}(2)\}$. One can check that $\mathrm{SU}(J) \cong (\mathrm{SU}(2) \times \mathbb{Z}_4)/\mathbb{Z}_2$. The variables are $\xi \in H^1(M, \mathbb{Z}_2)$ and $\alpha = 1 + \alpha^{(2)} + \alpha^{(4)}$. The equations under consideration are

$$\alpha^{(2)} = \beta_2(\xi) \tag{91}$$

$$\left(\alpha^{(2)}\right)^2 + 2\alpha^{(4)} = c_2(P). \tag{92}$$

Equation (91) fixes $\alpha^{(2)}$ in terms of ξ. For example, in the case $M = \mathrm{L}_p^3 \times \mathrm{S}^1$, by expanding $\xi = \xi_\mathrm{L}\, \gamma^{(1)}_{\mathrm{L}_p^3;\mathbb{Z}_2} \times 1_{\mathrm{S}^1} + \xi_\mathrm{S} 1_{\mathrm{L}_p^3;\mathbb{Z}_2} \times \gamma^{(1)}_{\mathrm{S}^1}$, equations (79) and (91) imply

$$\alpha^{(2)} = \begin{cases} q\xi_L\, \gamma^{(2)}_{\mathrm{L}_p^3;\mathbb{Z}} \times 1_{\mathrm{S}^1} & | \quad p = 2q \\ 0 & | \quad p = 2q + 1. \end{cases}$$

For general M, due to (91), equation (92) becomes $2\alpha^{(4)} = c_2(P)$. Thus, $\mathrm{K}(P, J)$ is nonempty iff $c_2(P)$ is even and is then parametrized by $\xi \in H^1(M, \mathbb{Z}_2)$.

6.4. Holonomy-induced Howe subbundles and factorization by SU(n)-action

In this subsection, we will accomplish steps 3 and 4 of our programme.

In step 3, we have to specify those reductions $Q \subseteq P$ to $\mathrm{SU}(J)$, $J \in \mathrm{K}(n)$, which are holonomy-induced, i.e. which possess a connected reduction to some subgroup H such that $\mathrm{C}^2_{\mathrm{SU}(n)}(H) = \mathrm{SU}(J)$. Let Q be given and consider a connected component of Q. This is a connected reduction of Q to some subgroup $H \subseteq \mathrm{SU}(J)$ which has the same dimension as $\mathrm{SU}(J)$. Then so has the Howe subgroup $\tilde{H} := \mathrm{C}^2_{\mathrm{SU}(n)}(H)$ generated by H, because $H \subseteq \tilde{H} \subseteq \mathrm{SU}(J)$. Then the Howe subgroups $\mathrm{C}^2_{\mathrm{U}(n)}(\tilde{H})$ and $\mathrm{C}^2_{\mathrm{U}(n)}(\mathrm{SU}(J))$ of U(n) have the same dimension and obey $\mathrm{C}^2_{\mathrm{U}(n)}(\tilde{H}) \subseteq \mathrm{C}^2_{\mathrm{U}(n)}(\mathrm{SU}(J))$. Since they are closed and connected (recall that they are conjugate to $\mathrm{U}(J)$ for some $J \in \mathrm{K}(n)$), they coincide. It follows $\tilde{H} = \mathrm{SU}(J)$. We conclude that any Howe subbundle of an SU(n)-bundle is holonomy-induced, so that the condition is redundant here.

We remark that, in general, Howe subbundles exist which are not holonomy-induced. A simple example is provided by the Howe subgroup $H = \{1_3, \mathrm{diag}(-1, -1, 1)\}$ of SO(3). While the reduction $Q = M \times H \subseteq M \times \mathrm{SO}(3)$ is a Howe subbundle, any connected reduction of Q has the centre $\{1_3\}$ as its structure group, hence is a Howe subbundle itself. Thus, Q is not holonomy-induced.

In step 4, we have to factorize the set of Howe subbundles by the principal action of $\mathrm{SU}(n)$. That is, we have to identify elements L, L' of $\mathrm{K}(P)$ for which $D \in \mathrm{SU}(n)$ exists such that

$$Q_{L'} \cong Q_L \cdot D. \tag{93}$$

First, assume that such D exists. Then $\mathrm{SU}(J') = D^{-1}\mathrm{SU}(J)D$, hence $\mathrm{M}_{J'}(\mathbb{C}) = D^{-1}\mathrm{M}_J(\mathbb{C})D$. It follows that $r = r'$ and there exists a permutation σ such that

$$\mathbf{k}' = \sigma\mathbf{k} \qquad \mathbf{m}' = \sigma\mathbf{m}. \tag{94}$$

A straightforward calculation, see [65, lemma 7.1] yields

$$\alpha_{J'}(Q_L \cdot D) = \sigma\alpha \qquad \xi_{J'}(Q_L \cdot D) = \xi. \tag{95}$$

Hence, (93) implies

$$\alpha' = \sigma\alpha \qquad \xi' = \xi. \tag{96}$$

Conversely, assume that $r = r'$ and that a permutation exists such that (94) and (96) hold. Due to (94) one can construct $D \in \mathrm{SU}(n)$ such that conjugation of $\mathrm{M}_J(\mathbb{C})$ by D^{-1} yields $\mathrm{M}_{J'}(\mathbb{C})$, where the factors are permuted according to σ, see [65, lemma 4.2]. Then (95) and (96) imply $\alpha_{J'}(Q_L \cdot D) = \alpha'$ and $\xi_{J'}(Q_L \cdot D) = \xi'$. Hence, (93) holds. Note that (93) is actually a special case of a more general situation discussed in subsection 7.1.

Thus, on the level of $\mathrm{K}(P)$, factorization by the principal $\mathrm{SU}(n)$-action on Howe subbundles amounts to the identification of elements which can be transformed to each other by a simultaneous permutation of $\mathbf{k}$, $\mathbf{m}$ and α. The set of equivalence classes so obtained will be denoted by $\hat{\mathrm{K}}(P)$ and its elements will be denoted by $[L]$.

6.5. Summary

Before we proceed, we summarize the results of this section. The set of Howe subbundles of P modulo isomorphy and the principal $\mathrm{SU}(n)$-action, which classifies the orbit types of the action of $\mathcal{G}$ on $\mathcal{C}$ by virtue of the reduction theorem, can be described as follows. Its elements are labelled by symbols $[J; \alpha, \xi]$, where

(i) $J = ((k_1, \ldots, k_r), (m_1, \ldots, m_r))$ is a pair of sequences of positive integers obeying $\sum_{i=1}^r k_i m_i = n$,

(ii) $\alpha = (\alpha_1, \ldots, \alpha_r)$ is a sequence of cohomology elements $\alpha_i \in H^*(M, \mathbb{Z})$, which are admissible values of the total Chern class of $\mathrm{U}(k_i)$-bundles over M,

(iii) $\xi \in H^1(M, \mathbb{Z}_g)$ with g being the greatest common divisor of $(m_1, \ldots, m_r)$. The cohomology elements α_i and ξ are subject to the relations

$$\sum_{i=1}^r \frac{m_i}{g}\alpha_i^{(2)} = \beta_g(\xi)$$

$$\alpha_1^{m_1} \ldots \alpha_r^{m_r} = c(P)$$

where $\beta_g : H^1(M, \mathbb{Z}_g) \to H^2(M, \mathbb{Z})$ is the connecting homomorphism associated with the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_g \to 0$ of coefficient groups in cohomology. For any permutation σ of r elements, the symbols

$$[((k_1, \ldots, k_r), (m_1, \ldots, m_r)) ; (\alpha_1, \ldots, \alpha_r), \xi]$$
$$\left[\left(\left(k_{\sigma(1)}, \ldots, k_{\sigma(r)}\right), \left(m_{\sigma(1)}, \ldots, m_{\sigma(r)}\right)\right) ; \left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)}\right), \xi\right]$$

have to be identified.

7. Partial ordering of gauge orbit types for $G = \mathrm{SU}(n)$

7.1. Characterization of the partial ordering relation

In this subsection we are going to characterize the natural partial ordering of Howe subbundles in terms of the classifying set $\hat{\mathrm{K}}(P)$. Thus, let $L, L' \in \mathrm{K}(P)$. By definition, $[L] \leqslant [L']$ iff $D \in \mathrm{SU}(n)$ exists such that $Q_L \cdot D \subseteq Q_{L'}$, where inclusion is understood up to isomorphy. We say that Q_L is subconjugate to $Q_{L'}$.

First, we observe that $Q_L \cdot D \subseteq Q_{L'}$ implies $D^{-1}\mathrm{SU}(J)D \subseteq \mathrm{SU}(J')$, i.e. subconjugacy of the structure groups. Then also $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$. We have an associated embedding

$$h_D^{\mathrm{M}} : \mathrm{M}_J(\mathbb{C}) \longrightarrow \mathrm{M}_{J'}(\mathbb{C}) \qquad C \mapsto D^{-1}CD$$

and, derived from that, embeddings $h_D^{\mathrm{U}} : \mathrm{U}(J) \longrightarrow \mathrm{U}(J')$ and $h_D^{\mathrm{S}} : \mathrm{SU}(J) \longrightarrow \mathrm{SU}(J')$. Since $\mathrm{M}_J(\mathbb{C})$ and $\mathrm{M}_{J'}(\mathbb{C})$ are finite-dimensional unital C*-algebras, the embedding h_D^{M} is characterized by a so-called *inclusion matrix* Δ. This is an $(r' \times r)$-matrix with nonnegative integer entries, defined as follows: $\Delta_{i'i}$ is the number of fundamental irreducible representations contained in the representation

$$\mathrm{M}_{k_i}(\mathbb{C}) \longrightarrow \mathrm{M}_J(\mathbb{C}) \xrightarrow{h_D^{\mathrm{M}}} \mathrm{M}_{J'}(\mathbb{C}) \xrightarrow{\mathrm{pr}^{\mathrm{M}}_{J',i'}} \mathrm{M}_{k'_{i'}}(\mathbb{C}).$$

Here the first map is the canonical embedding to the ith factor of $\mathrm{M}_J(\mathbb{C})$. Since the embedding h_D^{M} is unital, $\sum_i \Delta_{i'i}k_i = k'_{i'}$, for all i'. Since conjugation of $\mathrm{M}_J(\mathbb{C})$ by D^{-1} preserves the total number of fundamental irreducible representations of the factor $\mathrm{M}_{k_i}(\mathbb{C})$ in $\mathrm{M}_n(\mathbb{C})$, $\sum_{i'} \Delta_{i'i}m'_{i'} = m_i$, for all i. Thus, Δ solves the system of equations

$$\Delta\mathbf{k} = \mathbf{k}' \tag{97}$$

$$\mathbf{m} = \mathbf{m}'\Delta \tag{98}$$

where $\mathbf{m}$ and $\mathbf{m}'$ are viewed as row vectors. Conversely, assume that a solution Δ of (97) and (98) is given. Then the decompositions (55) associated with J and J' admit subdecompositions

$$\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \left(\bigoplus_{i'=1}^{r'} \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}} \right)$$

$$\mathbb{C}^n = \bigoplus_{i'=1}^{r'} \left(\bigoplus_{i=1}^{r} \mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \right) \otimes \mathbb{C}^{m'_{i'}}$$

respectively, which differ by a permutation of the factors $\mathbb{C}^{k_i} \otimes \mathbb{C}^{\Delta_{i'i}} \otimes \mathbb{C}^{m'_{i'}}$. From this permutation, $D \in \mathrm{SU}(n)$ can be constructed which obeys $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$ and which has inclusion matrix Δ, see [66, lemma 3.1]. It follows that $\mathrm{SU}(J)$ is subconjugate to $\mathrm{SU}(J')$, or $\mathrm{M}_J(\mathbb{C})$ is subconjugate to $\mathrm{M}_{J'}(\mathbb{C})$, iff the system of equations (97), (98) has a solution Δ.

Second, let Q_L^D denote the extension of $Q_L \cdot D$ to structure group $\mathrm{SU}(J')$. We observe that $Q_L \cdot D \subseteq Q_{L'}$ implies $Q_L^D \cong Q_{L'}$. This provides a relation between the characteristic classes α, ξ and α', ξ'. To derive it, we have to compute the characteristic classes of Q_L^D. Let us sketch how this can be done. For a detailed computation, purely on the level of cohomology, we refer to [66, lemma 3.2].

To compute $\alpha_{J'}\big(Q_L^D\big)$ we may form the extension $\widetilde{Q_L^D}$ of Q_L^D to structure group $\mathrm{U}(J')$ and compute the total Chern class of the Whitney factors. To do so, we use that $\widetilde{Q_L^D}$ coincides with the extension $\tilde{Q}_L^D$ of $\tilde{Q}_L \cdot D$ to structure group $\mathrm{U}(J')$. A close look at how h_D^{U} embeds the factors of $\mathrm{U}(J)$ into those of $\mathrm{U}(J)'$ reveals that the i'th Whitney factor of $\tilde{Q}_L^D$ contains the

Whitney product $(\tilde{Q}_L)_1^{\Delta_{i'1}} \times \cdots \times (\tilde{Q}_L)_r^{\Delta_{i'r}}$ as a subbundle. Hence, the total Chern class of this factor is $\alpha_1^{\Delta_{i'1}} \cdots \alpha_r^{\Delta_{i'r}}$. Using the notation

$$E_\Delta(\alpha) = \left(\alpha_1^{\Delta_{11}} \cdots \alpha_r^{\Delta_{1r}}, \ldots, \alpha_1^{\Delta_{r'1}} \cdots \alpha_r^{\Delta_{r'r}}\right)$$

which is a generalization of (69), we can write

$$\alpha_{J'}\left(Q_L^D\right) = E_\Delta(\alpha). \tag{99}$$

To determine $\xi_{J'}\left(Q_L^D\right)$, we can compute the class $\chi_{g'}$ of the quotient $Q_L^D \big/ \mathrm{SU}(J')_0$. The latter is given by the associated bundle $Q_L \times_{\mathrm{SU}(J)} \mathbb{Z}_{g'}$, where $\mathrm{SU}(J)$ acts on $\mathbb{Z}_{g'}$ via the homomorphism $\lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_D$. A straightforward computation yields $\lambda^{\mathrm{S}}_{J'} \circ h^{\mathrm{S}}_D = \varrho_{g'} \circ \lambda^{\mathrm{S}}_J$, where $\varrho_{g'}$ denotes reduction modulo g'. Note that (98) implies that g' divides g, hence $\varrho_{g'}$ is a well-defined homomorphism. Moreover, one can check that the characteristic class of the mod g'-reduction of a $\mathbb{Z}_g$-bundle is given by the mod g'-reduction of the characteristic class of this bundle. Hence

$$\xi_{J'}\left(Q_L^D\right) = \varrho_{g'}(\xi). \tag{100}$$

Thus, $Q_L \cdot D \subseteq Q_{L'}$ implies

$$E_\Delta(\alpha) = \alpha' \tag{101}$$

$$\varrho_{g'}(\xi) = \xi'. \tag{102}$$

Let us introduce the following notation. If (102) holds, let $\mathrm{N}(L, L')$ be the set of solutions of the combined system of equations (97), (98), (101) in the indeterminate Δ. If (102) does not hold, let $\mathrm{N}(L, L') = \emptyset$. So far, we have found that if Q_L is subconjugate to $Q_{L'}$ then $\mathrm{N}(L, L') \neq \emptyset$. Now assume that, conversely, $\mathrm{N}(L, L')$ contains an element Δ. We have seen above that due to (97), (98) there exists $D \in \mathrm{SU}(n)$, obeying $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$, which has inclusion matrix Δ. Consider Q_L^D, i.e. the extension of $Q_L \cdot D$ to structure group $\mathrm{SU}(J')$. Due to (99) and (101), $\alpha_{J'}\left(Q_L^D\right) = \alpha'$. Due to (100) and (102), $\xi_{J'}\left(Q_L^D\right) = \xi'$. It follows $Q_L^D \cong Q_{L'}$, hence $Q_L \cdot D \subseteq Q_{L'}$. Thus, we have shown that Q_L is subconjugate to $Q_{L'}$ iff $\mathrm{N}(L, L') \neq \emptyset$. Consequently, on the level of $\hat{\mathrm{K}}(P)$, the partial ordering of Howe subbundles is given by

Theorem 7.1. *Let $L, L' \in \mathrm{K}(P)$. Then $[L] \leqslant [L']$ if and only if $\mathrm{N}(L, L') \neq \emptyset$.*

Example. Let $P = M \times \mathrm{SU}(4)$. Consider elements L, L' with $J = ((1, 1), (2, 2))$ and $J' = ((2, 2), (1, 1))$, respectively. Recall that $\mathrm{SU}(J) \cong \mathrm{U}(1) \times \mathbb{Z}_2$. The subgroup $\mathrm{SU}(J')$ can be parametrized as follows:

$$\mathrm{SU}(J') = \left\{ \begin{pmatrix} zA & 0 \\ 0 & z^{-1}B \end{pmatrix} \middle| \, z \in \mathrm{U}(1), A, B \in \mathrm{SU}(2) \right\}.$$

It is therefore isomorphic to $[\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)]/\mathbb{Z}_2$. To determine $\mathrm{N}(L, L')$, we first consider equations (97) and (98):

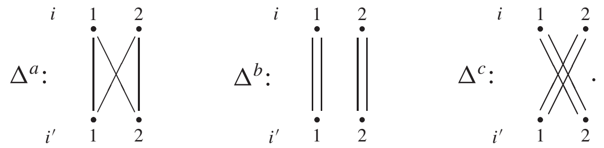
$$\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad (1 \quad 1) \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = (2 \quad 2).$$

The solutions are

$$\Delta^a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \Delta^b = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \Delta^c = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

For $\alpha = (\alpha_1, \alpha_2)$, they yield $E_{\Delta^a}(\alpha) = (\alpha_1\alpha_2, \alpha_1\alpha_2)$, $E_{\Delta^b}(\alpha) = \left(\alpha_1^2, \alpha_2^2\right)$, $E_{\Delta^c}(\alpha) = \left(\alpha_2^2, \alpha_1^2\right)$. Condition (102) is trivially satisfied due to $g' = 1$. Thus, $\mathrm{N}(L, L') \neq \emptyset$, i.e. Q_L is subconjugate to $Q_{L'}$ or $[L] \leqslant [L']$, precisely in one of the following cases: (a) $\alpha'_1 = \alpha'_2 = \alpha_1\alpha_2$, (b) $\alpha'_1 = \alpha_1^2, \alpha'_2 = \alpha_2^2$ and (c) $\alpha'_1 = \alpha_2^2, \alpha'_2 = \alpha_1^2$.

Remark. Any inclusion matrix can be visualized by a diagram consisting of a series of upper vertices, labelled by $i = 1, \ldots, r$, and a series of lower vertices, labelled by $i' = 1, \ldots, r'$. For each combination of i and i' the corresponding vertices are connected by $\Delta_{i'i}$ edges. For example, the matrices Δ^a, Δ^b, and Δ^c in the above example give rise to the following diagrams:

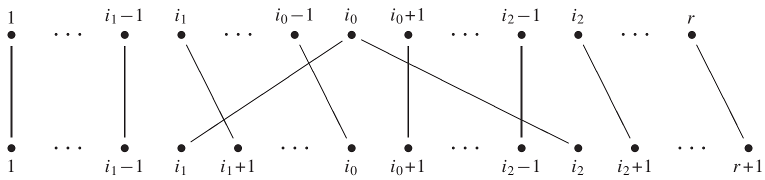


The diagrams associated in this way with the elements of $\mathrm{N}(J, J')$, $J, J' \in \mathrm{K}(n)$, are special cases of so-called *Bratteli diagrams* [18]. The latter have, in general, several stages picturing the subsequent inclusion matrices associated with an ascending sequence of finite dimensional von Neumann algebras $\mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \mathbf{A}_3 \subseteq \cdots$. For this reason, we refer to the diagram associated with $\Delta \in \mathrm{N}(J, J')$ as the Bratteli diagram of Δ. We remark that, due to equation (97), Δ cannot have a zero row. Due to (98), it cannot have a zero column either. Accordingly, each vertex of the Bratteli diagram of Δ is cut by at least one edge. Since equations (97), (98), (101) have an obvious reformulation on the level of Bratteli diagrams, these diagrams can be used to simplify calculations. Furthermore, some of the arguments in the sequel are easier to formulate on the level of Bratteli diagrams than on the level of the corresponding matrices.

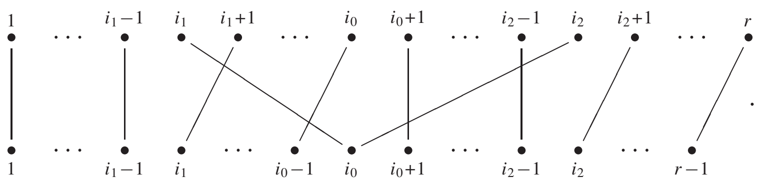
7.2. Direct successors

In this subsection we derive a characterization of direct successors. For a detailed discussion we refer to [66, section 5].

Let $L, L' \in \mathrm{K}(P)$ such that $[L] \leqslant [L']$. It is not hard to see that under this assumption $[L']$ is a direct successor of $[L]$ iff $[\mathrm{SU}(J')]$ is a direct successor of $[\mathrm{SU}(J)]$ in the set of conjugacy classes of Howe subgroups of $\mathrm{SU}(n)$, or iff $[\mathrm{M}_{J'}(\mathbb{C})]$ is a direct successor of $[\mathrm{M}_J(\mathbb{C})]$ in the set of conjugacy classes of unital $*$-subalgebras of $\mathrm{M}_n(\mathbb{C})$. It is known by 'folklore'—and can be proved using the notion of the level of an inclusion matrix, see [66]—that $[\mathrm{M}_{J'}(\mathbb{C})]$ is a direct successor of $[\mathrm{M}_J(\mathbb{C})]$ iff the following holds: there exists $D \in \mathrm{SU}(n)$ obeying $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$, where the Bratteli diagram of the corresponding inclusion matrix has either one of the following shapes with arbitrary i_0 and $i_1 < i_2$:



(103)



. (104)

Thus, if $[L']$ is a direct successor of $[L]$ then $\mathrm{N}(L, L')$ contains an element with Bratteli diagram (103) or (104). Conversely, if $\mathrm{N}(L, L')$ contains such an element Δ then $[L] \leqslant [L']$. As noted above, there exists $D \in \mathrm{SU}(n)$, obeying $D^{-1}\mathrm{M}_J(\mathbb{C})D \subseteq \mathrm{M}_{J'}(\mathbb{C})$, which has inclusion matrix Δ. Since the Bratteli diagram of Δ is of the form (103) or (104), $[\mathrm{M}_{J'}(\mathbb{C})]$ is a direct successor of $[\mathrm{M}_J(\mathbb{C})]$. Thus, $[L']$ is a direct successor of $[L]$. It follows

Theorem 7.2. *Let $L, L' \in \mathrm{K}(P)$. Then $[L']$ is a direct successor of $[L]$ if and only if $\mathrm{N}(L, L')$ contains an element with Bratteli diagram (103) or (104) for some i_0 and $i_1 < i_2$.*

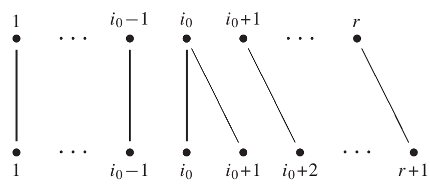
7.3. Generation of direct successors and direct predecessors

In this subsection, we sketch how to derive operations to create the direct successors and the direct predecessors of a given element of $\hat{\mathrm{K}}(P)$. Again, for a detailed discussion we refer to [66], sections 5 and 6.

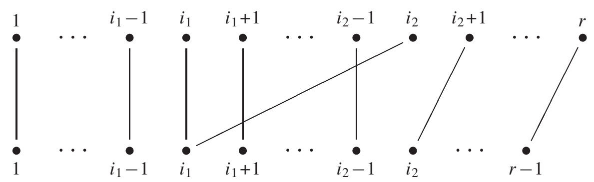
In view of theorem 7.2, to determine all direct successors of a given element $[L]$ of $\hat{\mathrm{K}}(P)$, we have to go through all the diagrams (103) and (104) and find all L' that obey (102) as well as the system of equations (97), (98), (101) with L being some representative of $[L]$ and Δ being given by the corresponding diagram. Of course, the amount of work can be reduced by observing that

(i) consideration of one representative L is sufficient,
(ii) diagrams that differ only by a permutation of the lower vertices yield equivalent L', hence identical direct successors.

It follows that the diagrams to be considered are



(105)



(106)

for arbitrary i_0 and $i_1 < i_2$, respectively. Taking this into account it can be easily seen that all necessary L' are generated from L by the following two kinds of operations:

Splitting: Choose i_0 such that $m_{i_0} \neq 1$. Choose a decomposition $m_{i_0} = m_{i_0,1} + m_{i_0,2}$ with strictly positive integers $m_{i_0,1}, m_{i_0,2}$. Define $J' = (\mathbf{k}', \mathbf{m}')$ and α' by

$$\begin{aligned}
\mathbf{k}' &= \left(k_1, \ldots, k_{i_0-1}, k_{i_0}, k_{i_0}, k_{i_0+1}, \ldots, k_r\right)\\
\mathbf{m}' &= \left(m_1, \ldots, m_{i_0-1}, m_{i_0,1}, m_{i_0,2}, m_{i_0+1}, \ldots, m_r\right)\\
\alpha' &= \left(\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0}, \alpha_{i_0}, \alpha_{i_0+1}, \ldots, \alpha_r\right).
\end{aligned}$$

Since the greatest common divisor g' of $\mathbf{m}'$ divides g, we can furthermore define $\xi' = \varrho_{g'}(\xi)$. We have to check whether $L' = (J'; \alpha', \xi')$ so defined is an element of

K(P). This can be done either by a direct computation or by the following argument. Due to $\mathbf{k}' \cdot \mathbf{m}' = n$, $J' \in \mathrm{K}(n)$. Moreover, L' solves the system of equations (97), (98), (101) with Δ being given by the Bratteli diagram (105). Thus, SU(J) is subconjugate to SU(J') by some $D \in \mathrm{SU}(n)$ with this inclusion matrix, and α' and ξ' are the characteristic classes of the extension Q_L^D of Q_L to structure group SU(J'). Hence, $L' \in \mathrm{K}(P)$. We say that L' arises from L by a splitting of the i_0th member.

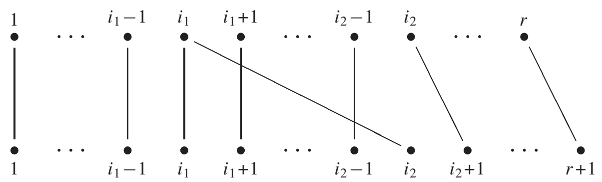
Merging: Choose $i_1 < i_2$ such that $m_{i_1} = m_{i_2}$. Define $J' = (\mathbf{k}', \mathbf{m}')$ and α' by

$$\begin{aligned}
\mathbf{k}' &= \left(k_1, \ldots, k_{i_1-1}, k_{i_1} + k_{i_2}, k_{i_1+1}, \ldots, \widehat{k_{i_2}}, \ldots, k_r\right) \\
\mathbf{m}' &= \left(m_1, \ldots, m_{i_1-1}, m_{i_1}, m_{i_1+1}, \ldots, \widehat{m_{i_2}}, \ldots, m_r\right) \\
\alpha' &= \left(\alpha_1, \ldots, \alpha_{i_1-1}, \alpha_{i_1}\alpha_{i_2}, \alpha_{i_1+1}, \ldots, \widehat{\alpha_{i_2}}, \ldots, \alpha_r\right)
\end{aligned}$$

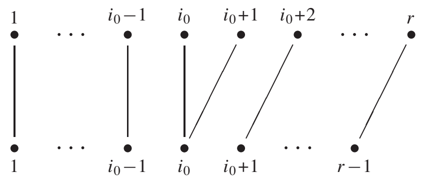
where '$\widehat{\ }$' indicates that the entry is omitted, as well as $\xi' = \xi$. To check that $L' = (J'; \alpha', \xi') \in \mathrm{K}(P)$ we proceed analogously to the case of splitting. We say that L' arises from L by merging the i_1th and the i_2th member.

We remark that it may happen that for certain elements of K(P) no splittings or no mergings can be applied. Amongst these elements are, for example, those with $m_1 = \cdots = m_r = 1$ (no splitting) and those having pairwise distinct m_i (no merging).

Next, we derive operations to create the direct predecessors of $[L]$. Direct predecessors are necessary to construct $\hat{\mathrm{K}}(P)$ from the unique maximal element (which is given by P itself). Note that predecessors correspond to strata of higher symmetry. Similar to the situation above, in view of theorem 7.2, we have to go through all the diagrams (103) and (104) and find all $L' \in \mathrm{K}(P)$ that obey (102) and the system of equations (97), (98), (101)—where L and L' have to be interchanged—with L being a representative of $[L]$ and Δ being given by the corresponding diagram. Again, we can reduce this work by noting that it suffices to consider a fixed representative L and by ignoring permutations, now of the upper vertices. The remaining diagrams to be considered are



(107)



(108)

with arbitrary $i_1 < i_2$ and i_0, respectively. One can check that all necessary L' are obtained by the following two kinds of operations, applied to L:

Inverse splitting: Choose $i_1 < i_2$ such that $k_{i_1} = k_{i_2}$ and $\alpha_{i_1} = \alpha_{i_2}$. Define $J' = (\mathbf{k}', \mathbf{m}')$ and α' by

$$\begin{aligned}
\mathbf{k}' &= \left(k_1, \ldots, k_{i_1-1}, k_{i_1}, k_{i_1+1}, \ldots, \widehat{k_{i_2}}, \ldots, k_r\right) \\
\mathbf{m}' &= \left(m_1, \ldots, m_{i_1-1}, m_{i_1} + m_{i_2}, m_{i_1+1}, \ldots, \widehat{m_{i_2}}, \ldots, m_r\right) \\
\alpha' &= \left(\alpha_1, \ldots, \alpha_{i_1-1}, \alpha_{i_1}, \alpha_{i_1+1}, \ldots, \widehat{\alpha_{i_2}}, \ldots, \alpha_r\right).
\end{aligned}$$

Then g divides the greatest common divisor g' of $\mathbf{m}'$, so that ϱ_g is well-defined. Choose $\xi' \in H^1(M, \mathbb{Z}_{g'})$ such that $\xi = \varrho_g(\xi')$ and $\beta_{g'}(\xi') = E^{(2)}_{\widetilde{\mathbf{m}}'}(\alpha')$. By construction, $L' = (J'; \alpha', \xi')$ is an element of $\mathrm{K}(P)$. We say that it arises from L by an inverse splitting of the i_1th and the i_2th members.

Inverse merging: Choose i_0 such that $k_{i_0} \neq 1$. Choose a decomposition $k_{i_0} = k_{i_0,1} + k_{i_0,2}$ with strictly positive integers $k_{i_0,1}, k_{i_0,2}$. Choose cohomology elements $\alpha_{i_0,1}, \alpha_{i_0,2} \in H^{\text{even}}(M, \mathbb{Z})$ such that $\alpha^{(2j)}_{i_0,l} = 0$ for $j > k_{i_0,l}$, $l = 1, 2$, and $\alpha_{i_0,1}\alpha_{i_0,2} = \alpha_{i_0}$. Define $J' = (\mathbf{k}', \mathbf{m}')$ and α' by

$$\begin{aligned}
\mathbf{k}' &= \big(k_1, \ldots, k_{i_0-1}, k_{i_0,1}, k_{i_0,2}, k_{i_0+1}, \ldots, k_r\big)\\
\mathbf{m}' &= \big(m_1, \ldots, m_{i_0-1}, m_{i_0}, m_{i_0}, m_{i_0+1}, \ldots, m_r\big)\\
\alpha' &= \big(\alpha_1, \ldots, \alpha_{i_0-1}, \alpha_{i_0,1}, \alpha_{i_0,2}, \alpha_{i_0+1}, \ldots, \alpha_r\big)
\end{aligned}$$

and $\xi' = \xi$. Again, by construction, $L' = (J'; \alpha', \xi') \in \mathrm{K}(P)$. We say that L' arises from L by an inverse merging of the i_0th member.

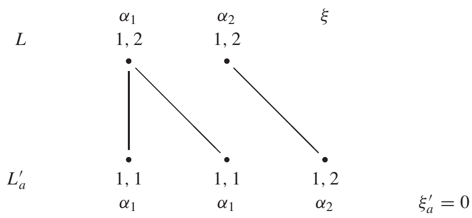
Let us summarize.

Theorem 7.3. *Let $[L] \in \hat{\mathrm{K}}(P)$ and let L be a representative. The direct successors (predecessors) of $[L]$ are obtained by applying all possible splittings and mergings (inverse splittings and inverse mergings) to L and passing to equivalence classes.*

7.4. Examples

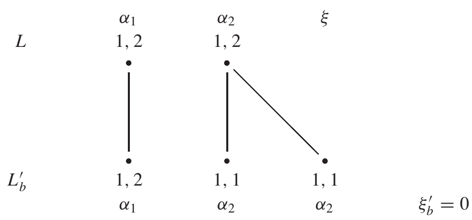
In this subsection, let P be a principal SU(4)-bundle.

Example 1. Direct successors of $[L]$ for $J = (1, 1|2, 2)$. (Recall the notation from subsection 6.3.) Note that α has components $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2$. Let us start with splitting operations. For $i_0 = 1$, the only possible splitting is given by the decomposition $m_1 = 2 = 1 + 1$. It yields $L'_a = (J'_a; \alpha'_a, \xi'_a)$, where $J'_a = (1, 1, 1|1, 1, 2)$, $\alpha'_a = (\alpha_1, \alpha_1, \alpha_2)$, and $\xi'_a = 0$. The passage from L to L'_a can very easily be performed on the level of a Bratteli diagram whose vertices are labelled by the respective quantities k_i, m_i and α_i (rather than by the mere number i):

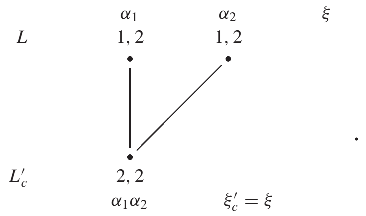


.

For $i_0 = 2$, a similar splitting operation creates L'_b, given by the labelled Bratteli diagram



As for merging operations, the only choice for i_1, i_2 is $i_1 = 1, i_2 = 2$. This yields L'_c:

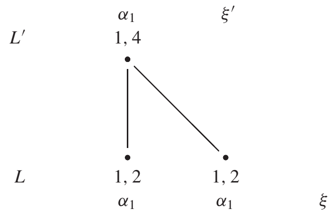


Next, we have to pass to equivalence classes. Generically, L'_a, L'_b, L'_c generate their own classes. However, while L'_c can never be equivalent to L'_a or L'_b, the latter are equivalent iff $\alpha_1 = \alpha_2$. In order to see for which bundle classes P this can happen, consider equations (77) and (78). The first one requires $\alpha_1^{(2)} = \alpha_2^{(2)}$ to be a torsion element. Then, due to $\alpha_1^{(4)} = \alpha_2^{(4)} = 0$, the second one implies $c_2(P) = 0$. Thus, L'_a and L'_b can be (occasionally) equivalent only if P is trivial.

Example 2. Direct predecessors of $[L]$ for $J = (1, 1|2, 2)$. Inverse splittings can be applied only if $\alpha_1 = \alpha_2$. In this case, for any solution $\xi \in H^1(M, \mathbb{Z}_4)$ of the system of equations

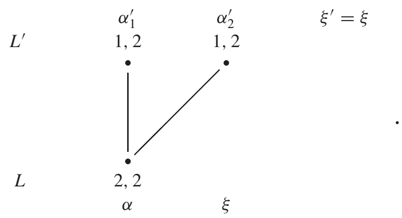
$$\xi' \bmod 2 = \xi \tag{109}$$
$$\beta_4(\xi') = \alpha_1^{(2)} \tag{110}$$

we obtain an element $L' = (J'; \alpha', \xi')$, where $J' = (1|4)$ and $\alpha' = \alpha_1 = \alpha_2$. The passage from L to L' can be summarized in the labelled Bratteli diagram



that has to be read upwards. Each L' generates its own equivalence class. Due to $k_1 = k_2 = 1$, inverse mergings cannot be applied to L. Thus, in the case $\alpha_1 = \alpha_2$ the direct predecessors of the equivalence class of L are labelled by the solutions of equations (109) and (110), whereas in the case $\alpha_1 \neq \alpha_2$ direct predecessors do not exist. Recall that the first case can only occur if P is trivial.

Example 3. Direct predecessors of $[L]$ for $J = (2|2)$. Here $\alpha = 1 + \alpha^{(2)} + \alpha^{(4)}$. Inverse mergings can be applied and yield elements L' as follows:



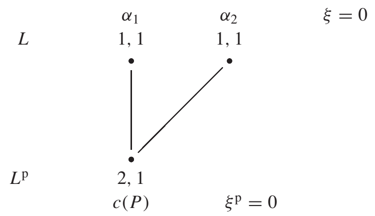
Here $\alpha'_i = 1 + \alpha_i'^{(2)}, i = 1, 2$, such that $\alpha'_1\alpha'_2 = \alpha$. When passing to equivalence classes, elements L' with (α'_1, α'_2) and (α'_2, α'_1) have to be identified. Since L does not allow for inverse splittings, there are no more direct predecessors.

8. Application

8.1. The stratification for SU(2)

In subsection 6.3 we have discussed particular examples of orbit types. In the present section we explain how to construct the Hasse diagram of the whole set of orbit types, starting from its maximal element. We restrict our attention to the simplest nontrivial case, the gauge group SU(2). We start with simple examples of base manifolds, for which the orbit types are known, and proceed to more complicated ones, such as lens spaces. This is intended to illustrate the technique. On the other hand, the means provided in subsection 7.3 enable us to construct the Hasse diagram for any SU(n). For SU(4), this was partially demonstrated in subsection 7.4. However, to present full Hasse diagrams for SU(4), or any other SU(n), in a transparent way needs some special graphical effort.

Let L^{p} denote the unique representative of the maximal element of $\hat{\mathrm{K}}(P)$. Since $Q_{L^{\mathrm{p}}} = P, L^{\mathrm{p}}$ is given by $J^{\mathrm{p}} = (2|1), \alpha^{\mathrm{p}} = c(P)$ and $\xi^{\mathrm{p}} = 0$. Inverse mergings yield elements L:



where $\alpha_i = 1 + \alpha_i^{(2)}$ such that $\alpha_1\alpha_2 = c(P)$. Sorting by degree yields the equations $\alpha_1^{(2)} + \alpha_2^{(2)} = 0$ and $\alpha_1^{(2)}\alpha_2^{(2)} = c_2(P)$. We obtain $\alpha_2^{(2)} = -\alpha_1^{(2)}$ and

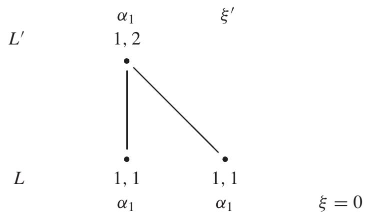
$$-\left(\alpha_1^{(2)}\right)^2 = c_2(P). \tag{111}$$

The solutions $\alpha_1^{(2)}$ and $-\alpha_1^{(2)}$ yield equivalent direct predecessors. We note that the Howe subgroup labelled by $J = (1, 1|1, 1)$ is the toral subgroup U(1) of SU(2) and that $\alpha_1^{(2)}$ is just the first Chern class of the corresponding reduction of P. By virtue of this transliteration, equation (111) is consistent with the literature [50].

Next, we determine the direct predecessors of the classes generated by L. Inverse mergings cannot be applied. Inverse splittings can be applied provided $\alpha_1 = \alpha_2$, i.e. $2\alpha_1^{(2)} = 0$. Then, for any solution $\xi' \in H^1(M, \mathbb{Z}_2)$ of the equation

$$\beta_2(\xi') = \alpha_1^{(2)}, \tag{112}$$

inverse merging yields an element L' by



.

Each of these elements generates its own equivalence class. Recall that $J = (1|2)$ labels the centre $\mathbb{Z}_2$ of SU(2) and that ξ' is the natural characteristic class for principal $\mathbb{Z}_2$-bundles over M.

Now let us draw Hasse diagrams of $\hat{\mathrm{K}}(P)$ for base manifolds $M = \mathrm{S}^4, \mathrm{S}^2 \times \mathrm{S}^2, \mathrm{L}^3_{2p} \times \mathrm{S}^1$. In the following, vertices stand for the elements of $\hat{\mathrm{K}}(P)$ and edges indicate the relation 'left vertex $\leqslant$ right vertex'. When viewing the elements of $\hat{\mathrm{K}}(P)$ as Howe subbundles, the vertex on the rhs represents the class corresponding to P itself, the vertices in the middle and on the lhs represent reductions of P to the Howe subgroups $\mathrm{U}(1)$ and $\mathbb{Z}_2$, respectively. When viewing the elements of $\hat{\mathrm{K}}(P)$ as orbit types, or strata of the gauge orbit space, the vertex on the rhs represents the generic stratum, whereas the vertices in the middle and on the lhs represent U(1)-strata and SU(2)-strata (the names refer to the isomorphy type of the corresponding stabilizer).

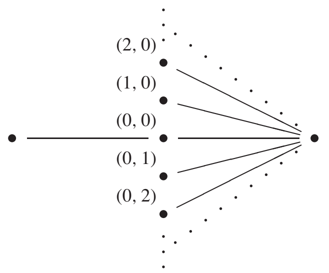
Example 1. $M = \mathrm{S}^4$. If $c_2(P) = 0$, equation (111) is trivially satisfied by $\alpha_1^{(2)} = 0$. Then equation (112) is trivially satisfied by $\xi' = 0$. Due to $H^1(M, \mathbb{Z}_2) = 0$ and $H^2(M, \mathbb{Z}) = 0$, there are no more solutions for either one. Thus, in the case where P is trivial, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is

$$\bullet\text{———}\bullet\text{———}\bullet$$

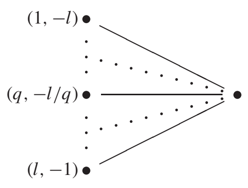
This situation was studied in detail, for instance, in [38].
If P is nontrivial, $\hat{\mathrm{K}}(P)$ is trivial, i.e. it consists only of the class corresponding to P itself.

On the level of strata, the result means that in the sector of vanishing topological charge the gauge orbit space decomposes into the generic stratum, a U(1)-stratum, and a SU(2)-stratum. If, on the other hand, a topological charge is present, only the generic stratum survives.

Example 2. $M = \mathrm{S}^2 \times \mathrm{S}^2$. Using the notation introduced in example 3 (iii) of subsection 6.3, equation (111) becomes $-2ab\, \gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2} = c_2(P)$. The discussion is similar to that of equation (86). Due to $H^1(M, \mathbb{Z}_2) = 0$, only the solution $a = b = 0$ has a direct predecessor itself. Thus, in the case $c_2(P) = 0$ the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



The vertices in the middle are labelled by the corresponding values of (a, b). Note that passage to equivalence classes requires identification of solutions (a, b) and $(-a, -b)$. In the case $c_2(P) = 2l\gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2}$, the Hasse diagram is



where, due to the identification $(a, b) \sim (-a, -b)$, q runs through the positive divisors of l only. Finally, in the case $c_2(P) = (2l + 1)\gamma^{(2)}_{\mathrm{S}^2} \times \gamma^{(2)}_{\mathrm{S}^2}$, $\hat{\mathrm{K}}(P)$ is trivial.

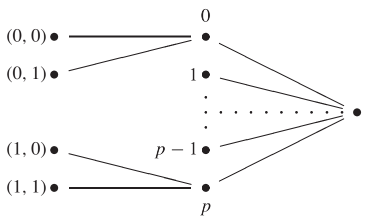
The interpretation of the result in terms of strata of the gauge orbit space is similar to that for spacetime manifold $M = \mathrm{S}^4$ above.

Example 3. $M = \mathrm{L}^3_{2p} \times \mathrm{S}^1$. Recall the notation from subsection 6.3. We write

$$\alpha_1^{(2)} = a\gamma^{(2)}_{\mathrm{L}^3_{2p};\mathbb{Z}} \times 1_{\mathrm{S}^1}. \tag{113}$$

Due to $H^2\big(\mathrm{L}^3_{2p}, \mathbb{Z}\big) \cong \mathbb{Z}_{2p}$, $\big(\alpha_1^{(2)}\big)^2 = 0$. Hence, equation (111) is solvable iff $c_2(P) = 0$, in which case the solutions are given by $a \in \mathbb{Z}_{2p}$. Since when passing to equivalence classes we have to identify solutions a and $-a$, the direct predecessors are labelled by elements of $\mathbb{Z}_p$.

Next, decomposing $\xi' = \xi'_{\mathrm{L}}\gamma^{(1)}_{\mathrm{L}^3_{2p};\mathbb{Z}_2} \times 1_{\mathrm{S}^1} + \xi'_{\mathrm{S}} 1_{\mathrm{L}^3_{2p};\mathbb{Z}_2} \times \gamma^{(1)}_{\mathrm{S}^1}$ and using (79), equation (112) becomes $p\xi'_{\mathrm{L}} = a$. Thus, only the elements labelled by $a = 0$ and $a = p$ have direct predecessors. These are given by the values $\xi'_{\mathrm{L}} = 0, \xi'_{\mathrm{S}} = 0, 1$ and $\xi'_{\mathrm{L}} = 1, \xi'_{\mathrm{S}} = 0, 1$, respectively. As a result, in the case $c_2(P) = 0$, the Hasse diagram of $\hat{\mathrm{K}}(P)$ is



Here the vertices on the lhs are labelled by $(\xi'_{\mathrm{L}}, \xi'_{\mathrm{S}})$, whereas those in the middle are labelled by a. In the case $c_2(P) \neq 0$, $\hat{\mathrm{K}}(P)$ is trivial. Again, the interpretation in terms of strata of the gauge orbit space goes along the lines of example 1 above.

8.2. Kinematical quantum nodes in Yang–Mills–Chern–Simons theory

Following [7], we consider gauge theory on the trivial bundle $\tilde{P} = (\Sigma \times \mathbb{R}) \times \mathrm{SU}(n)$, where Σ is a Riemann surface, in the Hamiltonian approach. The action functional consists of the Yang–Mills and the Chern–Simons term,

$$S(\tilde{A}) = \frac{1}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{tr}(\tilde{F}_{\tilde{A}} \wedge *\tilde{F}_{\tilde{A}}) + \frac{\lambda}{8\pi}\int_{\Sigma\times\mathbb{R}} \mathrm{tr}\left(\tilde{A} \wedge \tilde{F}_{\tilde{A}} - \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A}\right)$$

where $\tilde{A} \in \tilde{\mathcal{C}}$, the space of W^k-connections in $\tilde{P}$, and $\tilde{F}_{\tilde{A}}$ denotes the curvature of $\tilde{A}$. The coupling λ takes integer values. By separating the time variable, we get the following Lagrangian

$$L(A, A_0, \dot{A}, \dot{A}_0) = \frac{1}{2}(\dot{A} - \nabla_A A_0, \dot{A} - \nabla_A A_0)_0 - \frac{1}{2}(F_A, F_A)_0 + \frac{\lambda}{4\pi}\{2(A_0, *F_A)_0 + (A, *\dot{A})_0\}.$$

Here, $A_0 \in W^k(M, \mathrm{su}(n))$, A is a W^k-connection form in the trivial bundle $P = \Sigma \times \mathrm{SU}(n)$ and $(\cdot,\cdot)_0$ denotes the L^2-scalar product of $\mathrm{su}(n)$-valued forms on M. As usual, we denote the space of W^k-connections in P by $\mathcal{C}$. Constraint analysis yields the Gauß law

$$\nabla^*_A\Pi - \frac{\lambda}{4\pi} * \mathrm{d}A = 0$$

where Π denotes the momentum conjugate to A. Performing canonical quantization one finds that physical states are given by functions $\psi : \mathcal{C} \to \mathbb{C}$ that are contained in the kernel of the Gauß law operator

$$\nabla^*_{\hat{A}}\frac{\delta}{\delta A} - \frac{\mathrm{i}\lambda}{4\pi} * \mathrm{d}\hat{A} \tag{114}$$

where $\hat{A}$ means the multiplication operator, i.e. $(\hat{A}\psi)(A') = A'\psi(A'), \forall A' \in \mathcal{C}$. On the physical states, the Hamiltonian is given by

$$H = -\frac{1}{2}\left(\frac{\delta}{\delta A} + \mathrm{i}\frac{\lambda}{4\pi} * A, \frac{\delta}{\delta A} + \mathrm{i}\frac{\lambda}{4\pi} * A\right)_0 + \frac{1}{2}(F_A, F_A)_0.$$

Let us consider connections $A \in \mathcal{C}$ that can be reduced to some subbundle of P with nontrivial first Chern class. That is, in the language of physics, A carries a nontrivial magnetic charge. Thus, it may be viewed as monopole-like, although it is not assumed to be a solution of the field equations. In [4] it was shown that if the Chern–Simons term is present, i.e. $\lambda \neq 0$, then $\psi(A) = 0$ for any such A and any physical state ψ. Therefore, such A are called *kinematical* quantum nodes. Note that for geometric reasons there also exist dynamical nodes which differ from state to state. Due to their monopole-like character, kinematical quantum nodes are expected to play a role in the confinement mechanism. In the following we shall show that being a node is a property of strata. For that purpose, we reformulate the result of [4] in our language.

Theorem 8.1. *Let $A \in \mathcal{C}$ have orbit type $[(J; \alpha, \xi)] \in \hat{\mathrm{K}}(P)$. If $\alpha_i^{(2)} \neq 0$ for some i then A is a kinematical quantum node, i.e. $\psi(A) = 0$ for all physical states ψ.*

We outline the proof, following [4]. Let $L = (J; \alpha, \xi)$. Since Σ is a compact orientable 2-manifold, $H^2(\Sigma, \mathbb{Z}) = \mathbb{Z}$. Let $\gamma^{(2)}$ be a generator. Then $\alpha_i^{(2)} = c_i\gamma^{(2)}$ for certain $c_i \in \mathbb{Z}$. Consider the following element of $\mathrm{u}(n)$:

$$\tilde{\phi} := \mathrm{i}\left[\left(\frac{c_1}{k_1}1_{k_1} \otimes 1_{m_1}\right) \oplus \cdots \oplus \left(\frac{c_r}{k_r}1_{k_r} \otimes 1_{m_r}\right)\right].$$

Due to $(\alpha, \xi) \in \mathrm{K}(P, J)$, $(m_1c_1 + \cdots + m_rc_r)\gamma^{(2)} = E_{\mathbf{m}}^{(2)}(\alpha) = 0$. It follows $\mathrm{tr}(\tilde{\phi}) = 0$, hence $\tilde{\phi} \in \mathrm{su}(n)$. By construction, $\tilde{\phi}$ is invariant under the adjoint action of the subgroup $\mathrm{SU}(J) \subseteq \mathrm{SU}(n)$. Thus, we can define an equivariant function $\phi : P \to \mathrm{su}(n)$ by assigning to any $q \in Q_L$ the constant value $\tilde{\phi}$ and extending equivariantly to P. By construction, $\nabla_A\phi = 0$. Consequently, for any state $\psi : \mathcal{C} \to \mathbb{C}$,

$$\left(\phi, \left(\nabla_{\hat{A}}^* \frac{\delta}{\delta A}\psi\right)(A)\right)_0 = \left(\phi, \nabla_A^*\left\{\left(\frac{\delta}{\delta A}\psi\right)(A)\right\}\right)_0 = 0.$$

For physical states, the Gauss law implies

$$(\phi, (*\mathrm{d}\hat{A}\psi)(A))_0 = (\phi, *\mathrm{d}A)_0\psi(A) = 0. \tag{115}$$

Using $\nabla_A\phi = 0$ and the structure equation $F_A = \mathrm{d}A + \frac{1}{2}[A, A]$, we obtain

$$(\phi, *\mathrm{d}A)_0\psi(A) = 2(\phi, *F_A)_0\psi(A). \tag{116}$$

Since A is reducible to Q_L (recall that Q_L contains a holonomy bundle of A), F_A has block structure $((F_A)_1 \otimes 1_{m_1}) \oplus \cdots \oplus ((F_A)_r \otimes 1_{m_r})$ with $(F_A)_j$ being $(k_j \times k_j)$-matrices. Thus, by construction of ϕ,

$$(\phi, *F_A)_0 = \int_\Sigma \mathrm{Tr}(\phi F_A) = \mathrm{i}\sum_{j=1}^{r}\frac{m_j}{k_j}c_j\int_\Sigma \mathrm{Tr}((F_A)_j). \tag{117}$$

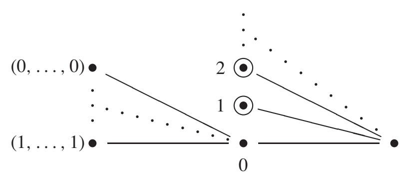
Since c_j are the first Chern classes of the Whitney factors of the extension of Q_L to structure group $\mathrm{U}(J) \cong \mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r)$, the integrals on the rhs give $-2\pi\mathrm{i}c_j$. Thus, equations (115)–(117) imply

$$\sum_{j=1}^{r}\frac{m_j}{k_j}c_j^2\psi(A) = 0. \tag{118}$$

It follows that if one of the c_j is nonzero then $\psi(A) = 0$, for all physical states ψ.

Remark. Let us compare (118) with formula (6) in [4]. Define $k_i' = k_i m_i$ and $m_i' = 1$. Then $J' = (\mathbf{k}', \mathbf{m}') \in \mathrm{K}(n)$ and $\mathrm{U}(J) \subseteq \mathrm{U}(J')$. Let Q_L' denote the extension of Q_L to structure group $\mathrm{U}(J')$. It is not hard to see that the Whitney factors of this subbundle have first Chern classes $c_i' = m_i c_i$. Inserting k_i', m_i', and c_i' into (118) one obtains formula (6) in [4]. In fact, the authors of [4] use that A is reducible to Q_L', rather than that it is even reducible to Q_L.

As a consequence of theorem 8.1, the property of being a kinematical node is actually a property of strata. It can be read off directly from the labels $L \in \mathrm{K}(P)$. As an example, we present the Hasse diagram of $\hat{\mathrm{K}}(P)$ for SU(2) (which can be derived analogously to the four-dimensional case explained in subsection 8.1), with the nodal strata marked by an additional circle:



The U(1)-strata are labelled by the moduli of the first Chern classes of the corresponding Q_L. The $\mathbb{Z}_2$-strata are labelled by elements of $\mathbb{Z}_2^{2s}$, where s is the genus of Σ. Thus, all but one U1-strata are kinematical nodes. The non-nodal stratum is that with zero topological charge. It is the only one which itself has singularities, where the singularities are all non-nodal.

9. Outlook

In the present review we have given a survey on the stratified structure of the gauge orbit space. Based on the results presented, a lot of points deserve a detailed study, for example

- the topology of strata, in particular w.r.t. potential anomalies [44],
- the geometric properties of strata w.r.t. the L^2-metric, in particular in the vicinity of singularities,
- the study of other metrics, like the strong metrics γ^k or η^k, defined in subsection 2 or the (potentially degenerate) information metric [37, 41].

From the viewpoint of physics, however, the most important question related to the stratified structure of the gauge orbit space is: what is the physical relevance of the nongeneric strata, i.e. what physical effects do they produce? To study this question systematically, one needs a quantization in which all strata are included on an equal footing and in which the stratification is explicitly encoded. To achieve this, we propose to view the gauge theory as an infinite-dimensional Hamiltonian system with symmetry and to work out the following programme:

1. Try to carry over the procedure of singular Marsden–Weinstein reduction, established in finite dimensions by Sjamaar and Lerman [72], to the infinite-dimensional Hamiltonian system under consideration (for an exposition of the method see [24, appendix B5] or [56, section IV.1.11]). Singular Marsden–Weinstein reduction equips the reduced phase space with the structure of a stratified symplectic space ('singular Marsden–Weinstein quotient'). A stratified symplectic space is a Poisson space X together with a stratification $X = \cup_i X_i$ (of some given type) into symplectic manifolds X_i such that the embeddings $X_i \to X$ are Poisson space morphisms.

2. Develop a geometric quantization of the reduced phase space so obtained. The generalization of methods of geometric quantization to stratified symplectic spaces is a field of active research. Besides the discussion of specific examples, until now the following notions have been established in finite dimensions:
   - prequantization of Poisson spaces [48] (applies to X),
   - prequantization of symplectic manifolds (standard, applies to the X_i),
   - polarization of stratified symplectic spaces [49].

Thus, to realize the concept of geometric quantization of a stratified symplectic space, the first problem to be solved consists in clarifying the relation between the prequantization of the Poisson space X and the prequantizations of its symplectic strata X_i. Next, using the above-mentioned polarization concept, one can try to construct the full quantum theory. Then, it is still a big challenge to extend these methods to the infinite-dimensional case.

Acknowledgments

The authors are indebted to M Čadek, T Friedrich, J Hilgert, C Isham, H-B Rademacher and L M Woodward for useful suggestions and remarks. IV is grateful to the Centre of Natural Sciences of the University of Leipzig for the warm hospitality extended to him during his stay in Leipzig.

Appendix A. Some basic facts from bundle theory

Classifying spaces and classifying maps. Let G be a Lie group. A principal G-bundle $E \to B$ is called universal for G, iff E is contractible. It can be shown that, for any Lie group G, there exists a universal bundle

$$G \hookrightarrow \mathrm{E}G \stackrel{\pi_G}{\to} \mathrm{B}G$$

with the following property: For any CW-complex (hence, in particular, any manifold) X the assignment

$$[X, \mathrm{B}G] \longrightarrow \mathrm{Bun}(X, G) \qquad f \mapsto f^*\mathrm{E}G \tag{A.1}$$

is a bijection. Here, $[\cdot, \cdot]$ denotes the set of homotopy classes of maps, $\mathrm{Bun}(X, G)$ is the set of isomorphism classes of principal G-bundles over X (where bundle morphisms are assumed to project to the identical mapping on X) and f^* denotes the pull-back of bundles: $f^*\mathrm{E}G = \{(x, \epsilon) \in X \times \mathrm{E}G : f(x) = \pi_G(\epsilon)\}$. $\mathrm{B}G$ is called the classifying space of G and the homotopy class of maps $X \to \mathrm{B}G$ associated with $P \in \mathrm{Bun}(X, G)$ by virtue of (A.1) is called the classifying map of P. In this appendix, we will denote it by f_P. Since the total space of $\mathrm{E}G$ is contractible, the exact homotopy sequence of fibre spaces implies

$$\pi_i(G) \cong \pi_{i+1}(\mathrm{B}G) \qquad i = 0, 1, 2, \ldots. \tag{A.2}$$

Associated principal bundles defined by homomorphisms. Let $\varphi : G \to G'$ be a Lie group homomorphism and let $P \in \mathrm{Bun}(X, G)$. By virtue of the action

$$G' \times G \to G' \qquad (a', a) \mapsto \varphi(a^{-1})a'$$

G' becomes a right G-space and we have an associated bundle $P \times_G G'$. To indicate that this bundle is completely given by φ, we denote it by $P^{[\varphi]}$. By setting $[(p, a')] \cdot b' := [(p, a'b')]$, $\forall p \in P, a', b' \in G'$, a right G'-action on $P^{[\varphi]}$ is defined, thus making it a principal G'-bundle over X.

In the main text of the review, two special cases of associated principal bundles occur:

(i) φ is a Lie subgroup embedding. Here $P^{[\varphi]}$ represents the extension of P to structure group G'.
(ii) φ is factorization by a normal Lie subgroup N. Here $P^{[\varphi]}$ represents the quotient bundle P/N.

Thus, the construction of associated principal bundle provides a unifying viewpoint on the operations of extending the structure group and factorizing by a normal subgroup. In particular, it allows to determine the classifying map of both extensions and quotients. Namely, one has

$$f_{P[\varphi]} = \mathrm{B}\varphi \circ f_P \tag{A.3}$$

where $\mathrm{B}\varphi : \mathrm{B}G \to \mathrm{B}G'$ is the map of classifying spaces associated with φ. It is defined as the classifying map of the associated principal G'-bundle $(\mathrm{E}G)^{[\varphi]}$. Note the following (covariant) functorial property: For $\varphi : G \to G'$ and $\varphi' : G' \to G''$ there holds

$$\mathrm{B}(\psi \circ \varphi) = \mathrm{B}\psi \circ \mathrm{B}\varphi.$$

Appendix B. Eilenberg–MacLane spaces and Postnikov tower

Eilenberg–MacLane spaces. Let π be a group and n a positive integer. An arcwise connected CW-complex X is called an Eilenberg–MacLane space of type $K(\pi, n)$ iff $\pi_n(X) = \pi$ and $\pi_i(X) = 0$ for $i \neq n$. Eilenberg–MacLane spaces exist for any choice of π and n, provided π is commutative for $n \geqslant 2$. They are unique up to homotopy equivalence.

The simplest example of an Eilenberg–MacLane space is the 1-sphere S^1, which is of type $K(\mathbb{Z}, 1)$. Two further examples, $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_g, 1)$, are briefly discussed in appendix C. Apart from very special examples, Eilenberg–MacLane spaces are infinite dimensional. Up to homotopy equivalence one has

$$K(\pi_1 \times \pi_2, n) = K(\pi_1, n) \times K(\pi_2, n).$$

Now assume π to be commutative also in the case $n = 1$. Due to the Hurewicz and the universal coefficient theorems, $H^n(K(\pi, n), \pi) = \mathrm{Hom}\,(H_n(K(\pi, n)), \pi)$. Moreover, $H_n(K(\pi, n)) \cong \pi_n(K(\pi, n)) = \pi$. It follows that $H^n(K(\pi, n), \pi)$ contains elements which correspond to isomorphisms $H_n(K(\pi, n)) \to \pi$. Such elements are called *characteristic*. If $\gamma \in H^n(K(\pi, n), \pi)$ is characteristic then for any CW-complex X, the map

$$[X, K(\pi, n)] \to H^n(X, \pi) \qquad f \mapsto f^*\gamma \tag{B.1}$$

is a bijection [20, section VII.12]. In this sense, Eilenberg–MacLane spaces provide a link between homotopy properties and cohomology.

Next, consider the path-loop fibration over $K(\pi, n)$,

$$\Omega(K(\pi, n)) \hookrightarrow P(K(\pi, n)) \longrightarrow K(\pi, n)$$

where $\Omega(K(\pi, n))$ and $P(K(\pi, n))$ denote the loop space and the path space of $K(\pi, n)$, respectively (both based at some point x_0). Since $P(K(\pi, n))$ is contractible, the exact homotopy sequence induced by the path-loop fibration implies $\pi_i\,(\Omega(K(\pi, n + 1))) = \pi_{i+1}\,(K(\pi, n + 1))$ Hence, $\Omega(K(\pi, n + 1)) = K(\pi, n)$, $\forall n$, and the path-loop fibration over $K(\pi, n + 1)$ reads

$$K(\pi, n) \hookrightarrow P(K(\pi, n + 1)) \longrightarrow K(\pi, n + 1). \tag{B.2}$$

Postnikov tower. A map $f : X \to X'$ of topological spaces is called an n-equivalence iff the homomorphism induced on homotopy groups $f_* : \pi_i(X) \to \pi_i(X')$ is an isomorphism for

$i < n$ and surjective for $i = n$. One also defines the notion of an ∞-equivalence, which is often called weak homotopy equivalence.

Let $f : X \to X'$ be an n-equivalence and let Y be a CW-complex. Then the map $[Y, X] \to [Y, X']$, $g \mapsto f \circ g$, is bijective for $\dim Y < n$ and surjective for $\dim Y = n$ [20, chapter VII, corollary 11.13].

A CW-complex Y is called n-simple iff it is arcwise connected and the action of $\pi_1(Y)$ on $\pi_i(Y)$ is trivial for $1 \leqslant i \leqslant n$. It is called simple iff it is n-simple for all n.

The following theorem describes how a simple CW-complex can be approximated by n-equivalent spaces constructed from Eilenberg–MacLane spaces.

Theorem B.1. *Let Y be a simple CW-complex. There exist*

(i) a sequence of CW-complexes Y_n and principal fibrations

$$K(\pi_n(Y), n) \hookrightarrow Y_{n+1} \xrightarrow{q_n} Y_n \qquad n = 1, 2, 3, \ldots$$

given as the pull-back of the path-loop fibration (B.2) over $K(\pi_n(Y), n+1)$ by some map $\theta_n : Y_n \to K(\pi_n(Y), n+1)$,

(ii) a sequence of n-equivalences $y_n : Y \to Y_n$, $n = 1, 2, 3, \ldots$,

such that $Y_1 = *$ (one point space) and $q_n \circ y_{n+1} = y_n$ for all n.

The sequence of spaces and maps (Y_n, y_n, q_n), $n = 1, 2, 3, \ldots$, is called a Postnikov tower (or Postnikov decomposition) of Y.

We remark that the theorem follows from a more general theorem about simple maps [20, chapter VII, theorem 13.7] by noting that the assumption that Y is a simple CW-complex implies that the constant map $Y \to *$ is a simple map. See [20, chapter VII, definition 13.4] for a definition of the latter.

Appendix C. Construction of BSU$(J)_5$

In this appendix, let $J \in \mathrm{K}(n)$ and consider the classifying space $\mathrm{BSU}(J)$ of the Howe subgroup $\mathrm{SU}(J)$. We are going to prove that $\mathrm{BSU}(J)_5$, i.e. the fifth stage of the Postnikov tower of $\mathrm{BSU}(J)$, is given by formula (59).

Preparation. First, in order to be able to apply theorem B.1, we have to check that $\mathrm{BSU}(J)$ is a simple space. To see this, note that any inner automorphism of $\mathrm{SU}(J)$ is generated by an element of $\mathrm{SU}(J)_0$, hence is homotopic to the identity automorphism. Consequently, the natural action of $\pi_0(\mathrm{SU}(J))$ on $\pi_{i-1}(\mathrm{SU}(J))$, $i = 1, 2, 3, \ldots$, induced by inner automorphisms, is trivial. Since the natural isomorphisms $\pi_{i-1}(\mathrm{SU}(J)) \cong \pi_i(\mathrm{BSU}(J))$ transform this action into that of $\pi_1(\mathrm{BSU}(J))$ on $\pi_i(\mathrm{BSU}(J))$, the latter is trivial, too. Thus, $\mathrm{BSU}(J)$ is a simple space, as asserted.

Second, we note the relevant homotopy groups of $\mathrm{BSU}(J)$. According to (58) and (A.2), these are

$$\pi_1 = \mathbb{Z}_g \qquad \pi_2 = \mathbb{Z}^{\oplus(r-1)} \qquad \pi_3 = 0 \qquad \pi_4 = \mathbb{Z}^{\oplus r^*} \tag{C.1}$$

where r^* denotes the number of $k_i > 1$.

Third, we will need information about the integer-valued cohomology groups of the Eilenberg–MacLane spaces $K(\mathbb{Z}_g, 1)$ and $K(\mathbb{Z}, 2)$.

(i) $K(\mathbb{Z}, 2)$: Consider the natural free action of $\mathrm{U}(1)$ on the sphere S^∞ (induced from $\mathrm{U}(1)$-action on $\mathrm{S}^{2n-1} \subseteq \mathbb{C}^n$). The orbit space of this action is known as the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$. Due to $\pi_i(\mathrm{S}^\infty) = 0$, the exact homotopy sequence of the principal bundle $\mathrm{U}(1) \hookrightarrow \mathrm{S}^\infty \to \mathbb{C}\mathrm{P}^\infty$ implies $\pi_i(\mathbb{C}\mathrm{P}^\infty) = \pi_{i-1}(\mathrm{U}(1)) = \mathbb{Z}$, for $i = 2$ and 0 otherwise. Thus, $\mathbb{C}\mathrm{P}^\infty$ is a model for $K(\mathbb{Z}, 2)$. It follows, see [20, chapter VI, proposition 10.2],

$$H^i(K(\mathbb{Z}, 2), \mathbb{Z}) = \begin{cases} \mathbb{Z} & | \quad i \text{ even} \\ 0 & | \quad i \text{ odd.} \end{cases} \tag{C.2}$$

(ii) $K(\mathbb{Z}_g, 1)$: Consider the restriction of the above action to the subgroup $\mathbb{Z}_g \subseteq \mathrm{U}(1)$. The resulting orbit space is the infinite-dimensional lens space L_g^∞. The exact homotopy sequence of the corresponding principal bundle implies $\pi_i\left(\mathrm{L}_g^\infty\right) = \pi_{i-1}(\mathbb{Z}_g) = \mathbb{Z}_g$, for $i = 1$ and 0 otherwise. Hence, L_g^∞ is a model for $K(\mathbb{Z}_g, 1)$. Consequently, see [34, section 24, p 176],

$$H^i(K(\mathbb{Z}_g, 1), \mathbb{Z}) = \begin{cases} \mathbb{Z} & | \quad i = 0 \\ \mathbb{Z}_g & | \quad i \neq 0, \text{even} \\ 0 & | \quad i \neq 0, \text{odd.} \end{cases} \tag{C.3}$$

(Note that the vanishing of all homotopy groups of S^∞ also implies that $\mathbb{C}\mathrm{P}^\infty$ and L_g^∞ are models for the classifying spaces $\mathrm{BU}(1)$ and $\mathrm{B}\mathbb{Z}_g$, respectively.)

Construction. We start with $\mathrm{BSU}(J)_1 = *$. Then $\mathrm{BSU}(J)_2$ must coincide with the fibre which is $K(\mathbb{Z}_g, 1)$, see (C.1). Next, according to (C.1), $\mathrm{BSU}(J)_3$ is the total space of a fibration

$$K\left(\mathbb{Z}^{\oplus(r-1)}, 2\right) \hookrightarrow \mathrm{BSU}(J)_3 \xrightarrow{q_2} K(\mathbb{Z}_g, 1) \tag{C.4}$$

given by the pull-back of the path-loop fibration over $K(\mathbb{Z}^{\oplus(r-1)}, 3)$ by some map $\theta_2 : K(\mathbb{Z}_g, 1) \to K(\mathbb{Z}^{\oplus(r-1)}, 3)$. Since $K(\mathbb{Z}^{\oplus(r-1)}, n) = \prod_{j=1}^{r-1} K(\mathbb{Z}, n), \forall n$, (B.1) yields for the set of homotopy classes

$$\left[K(\mathbb{Z}_g, 1), K\left(\mathbb{Z}^{\oplus(r-1)}, 3\right)\right] = \prod_{i=1}^{r-1} H^3(K(\mathbb{Z}_g, 1), \mathbb{Z}).$$

Due to (C.3), the rhs is trivial. Hence, θ_2 is homotopic to a constant map, so that the fibration (C.4) is trivial. Thus

$$\mathrm{BSU}(J)_3 = K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2).$$

Then, in view of (C.1), $\mathrm{BSU}(J)_4$ is given by a fibration over $\mathrm{BSU}(J)_3$ with fibre $K(0, 3) = *$. Hence, it just coincides with the base space. Finally, $\mathrm{BSU}(J)_5$ is the total space of a fibration

$$K(\mathbb{Z}^{\oplus r^*}, 4) \hookrightarrow \mathrm{BSU}(J)_5 \xrightarrow{q_4} K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2) \tag{C.5}$$

which is induced by a map θ_4 from the base to $K(\mathbb{Z}^{\oplus r^*}, 5)$. We have

$$\left[K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), K(\mathbb{Z}^{\oplus r^*}, 5)\right] = \prod_{i=1}^{r^*} H^5\left(K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), \mathbb{Z}\right).$$

Since $H^*(K(\mathbb{Z}, 2), \mathbb{Z})$ is torsion-free, see (C.2), we can apply the Künneth theorem for cohomology to obtain

$$\begin{aligned} H^5&\left(K(\mathbb{Z}_g, 1) \times \prod_{j=1}^{r-1} K(\mathbb{Z}, 2), \mathbb{Z}\right) \\ &\cong \bigoplus H^j(K(\mathbb{Z}_g, 1), \mathbb{Z}) \otimes H^{j_1}(K(\mathbb{Z}, 2), \mathbb{Z}) \otimes \cdots \otimes H^{j_{r-1}}(K(\mathbb{Z}, 2), \mathbb{Z}) \end{aligned}$$

with the direct sum running over all decompositions of 5 into a sum of r nonnegative integers $j, j_1, \ldots, j_{r-1}$. Each summand of the rhs is trivial, because it contains tensor factors of odd degree, which are trivial due to (C.2) and (C.3). Hence, θ_4 is again homotopic, a constant map and the fibration (C.5) is trivial. This proves formula (59), used in the main text.

References

[1] Abbati M C, Cirelli R, Manià A and Michor P 1989 The Lie group of automorphisms of a principal bundle *J. Geom. Phys.* **6** 215–35

[2] Abbati M C, Cirelli R and Manià A 1989 The orbit space of the action of gauge transformation group on connections *J. Geom. Phys.* **6** 537–57

[3] Alvarez-Gaume L and Ginsparg P 1984 The topological meaning of nonabelian anomalies *Nucl. Phys.* B **243** 449–74

[4] Asorey M, Falceto F, López J L and Luzón G 1995 Nodes, monopoles, and confinement in (2 + 1)-dimensional gauge theories *Phys. Lett.* B **345** 125–30

[5] Asorey M 1999 Maximal non-Abelian gauges and topology of the gauge orbit space *Nucl. Phys.* B **551** 399–424

[6] Asorey M and Mitter P K 1981 Regularized, continuum Yang–Mills process and Feynman–Kac functional integral *Commun. Math. Phys.* **80** 43–58

[7] Asorey M and Mitter P K 1985 Cohomology of the gauge orbit space and (2 + 1)-dimensional Yang–Mills theory with the Chern–Simons term *Phys. Lett.* B **153** 147–52

[8] Atiyah M F and Singer I M 1984 Dirac operators coupled to vector potentials *Proc. Natl Acad. Sci. USA* **81** 2597–600

[9] Avis S J and Isham C J 1979 Quantum field theory and fibre bundles in a general space-time *Recent Developments in Gravitation (Cargèse 1978)* ed M Levy and S Deser (New York: Plenum) pp 347–401

[10] Babelon O and Viallet C-M 1979 The geometrical interpretation of the Faddeev–Popov determinant *Phys. Lett.* B **85** 246–48

[11] Babelon O and Viallet C-M 1981 The Riemannian geometry of the configuration space of gauge theories *Commun. Math. Phys.* **81** 515–25

[12] Berzi V and Reni M 1987 Weak Riemannian structures one gauge-group orbits *Int. J. Theor. Phys.* **26** 151–74

[13] Borel A 1967 *Topics in the Homology Theory of Fibre Bundles (Lecture Notes in Mathematics vol 36)* (New York: Springer)

[14] Bourbaki N 1989 *Lie Groups and Lie Algebras* (Berlin: Springer) ch 1–3 and 9

[15] Bourbaki N 1967 *Variétés Différentielles et Analytiques, Fascicules des Résultats* (Paris: Hermann) ch 1–7

[16] Bourbaki N 1989 *General Topology* (New York: Springer)

[17] Bourguignon J 1975 Une stratification de l'espace des structures riemanniennes *Compositio Math.* **30** 1–41

[18] Bratteli O 1972 Inductive limits of finite dimensional C^*-algebras *Trans. Am. Math. Soc.* **171** 195–234

[19] Bredon G E 1972 *Introduction to Compact Transformation Groups* (New York: Academic)

[20] Bredon G E 1993 *Topology and Geometry* (Berlin: Springer)

[21] Cantor M 1981 Elliptic operators and the decomposition of tensor fields *Bull. Am. Math. Soc.* **5** 235–62

[22] Carey A, Mickelsson J and Murray M 1997 Index theory, gerbes, and Hamiltonian quantization *Commun. Math. Phys.* **183** 707–22

[23] Choquet-Bruhat Y and Christodoulou D 1981 Elliptic systems in Hilbert spaces on manifolds which are euclidean at infinity *Acta Math.* **146** 129–50

[24] Cushman R H and Bates L M 1997 *Global Aspects of Classical Integrable Systems* (Basel: Birkhäuser)

[25] Daniel M and Viallet C-M 1980 The geometrical setting of gauge theories of the Yang–Mills type *Rev. Mod. Phys.* **52** 175–97

[26] Dell'Antonio G and Zwanziger D 1991 Every gauge orbit passes inside the Gribov horizon *Commun. Math. Phys.* **138** 291–9

[27] Ebin D G 1970 The manifold of Riemannian metrics *Proc. Symp. Pure Math. Am. Math. Soc.* **XV** 11–40

[28] Eguchi T, Gilkey P B and Hanson A J 1980 Gravitation, gauge theories and differential geometry *Phys. Rep.* **66** 213–393

[29] Emmrich C and Römer H 1990 Orbifolds as configuration spaces of systems with gauge symmetries *Commun. Math. Phys.* **129** 69–94

[30] Faddeev L D and Popov V N 1967 Feynman diagrams for the Yang–Mills field *Phys. Lett.* B **25** 29–30

[31] Fischer A E 1970 The theory of superspace *Relativity (Proc. Conf. Midwest, Cincinnati, Ohio, 1969)* (New York: Plenum) pp 303–57

[32] Fischer A E 1987 The internal symmetry group of a connection on a principal fiber bundle with applications to gauge field theories *Commun. Math. Phys.* **113** 231–62

[33] Fleischhack Chr 2000 Stratification of the generalized gauge orbit space *Commun. Math. Phys.* **214** 607–49
[34] Fomenko A T, Fuchs D B and Gutenmacher V L 1986 *Homotopic Topology* (Budapest: Akadémiai Kiadó)
[35] Ford C, Tok T and Wipf A 1999 Abelian projection on the torus for general gauge groups *Nucl. Phys.* B **548** 585–612
[36] Ford C, Tok T and Wipf A 1999 SU(n)-gauge theories in Polyakov gauge on the torus *Phys. Lett.* B **456** 155–61
[37] Friedrich T 1991 Die Fisher-Information und symplektische Strukturen [Fisher information and symplectic structures] *Math. Nachr.* **153** 273–96 (in German)
[38] Fuchs J, Schmidt M G and Schweigert C 1994 On the configuration space of gauge theories *Nucl. Phys.* B **426** 107–28
[39] Gawedzki K 1982 Yang–Mills theory as Schrödinger quantum mechanics on the space of gauge-group orbits *Phys. Rev.* D **26** 3593–610
[40] Gribov V N 1978 Quantization of non-Abelian gauge theories *Nucl. Phys.* B **139** 1–19
[41] Groisser D and Murray M K 1997 Instantons and the information metric *Ann. Global Anal. Geom.* **15** 519–37
[42] Habermann L 1992 A family of metrics on the moduli space of $\mathbb{C}P^2$ instantons *Commun. Math. Phys.* **149** 209–16
[43] Heil A, Kersch A, Papadopoulos N A, Reifenhäuser B and Scheck F 1990 Structure of the space of reducible connections for Yang–Mills theories *J. Geom. Phys.* **7** 489–505
[44] Heil A, Kersch A, Papadopoulos N A, Reifenhäuser B and Scheck F 1990 Anomalies from nonfree action of the gauge group *Ann. Phys.* **200** 206–15
[45] Howe R 1979 θ-series and invariant theory *Automorphic Forms, Representations, and L-functions, Proc. Symp. Pure Math.* **33** pp 275–85
[46] Howe R 1985 Dual pairs in physics: harmonic oscillators, photons, electrons and singletons *Lect. Appl. Math.* vol 21 (Providence, RI: American Mathematical Society) pp 179–207
[47] Howe R 1989 Transcending classical invariant theory *J. Am. Math. Soc.* **2** 535–52
[48] Hübschmann J 1990 Poisson cohomology and quantization *J. Reine Angew. Math.* **408** 57–113
[49] Hübschmann J 2001 Kaehler spaces, nilpotent orbits, and singular reduction *Preprint* math-dg/0104213
[50] Isham C J 1981 Space-time topology and spontaneous symmetry breaking *J. Phys. A: Math. Gen.* **14** 2943–56
[51] Killingback T P 1984 The Gribov ambiguity in gauge theories on the four-torus *Phys. Lett.* B **138** 87–90
[52] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol I (New York: Wiley Interscience)
[53] Kondracki W and Rogulski J 1986 On the notion of stratification *Demonstratio Math.* **19** 229–36
[54] Kondracki W and Rogulski J 1986 On the stratification of the orbit space for the action of automorphisms on connections *Dissertationes Mathematicae* vol 250 (Warszawa: Panstwowe Wydawnictwo Naukowe)
[55] Kondracki W and Sadowski P 1986 Geometric structure on the orbit space of gauge connections *J. Geom. Phys.* **3** 421–33
[56] Landsman N P 1998 *Mathematical Topics between Classical and Quantum Mechanics* (New York: Springer)
[57] Langmann E, Salmhofer M and Kovner A 1994 Consistent axial-like gauge fixing on hypertori *Mod. Phys. Lett.* A **9** 2913–26
[58] Mitter P K 1980 Geometry of the gauge orbit space and the Yang–Mills dynamical system *Cargèse Lectures in Recent Developments in Gauge Theory* ed G t'Hooft *et al* (New York: Plenum)
[59] Mitter P K and Viallet C-M 1981 On the bundle of connections and the gauge orbit manifold in Yang–Mills theory *Commun. Math. Phys.* **79** 457–72
[60] Narasimhan M S and Ramadas T R 1979 Geometry of SU(2) gauge fields *Commun. Math. Phys.* **67** 121–36
[61] Palais R S 1968 *Foundations of Global Nonlinear Analysis* (New York: Benjamin)
[62] Przebinda T 1988 On Howe's duality theorem *J. Funct. Anal.* **81** 160–83
[63] Rogulski J S 1985 Operators with H^k coefficients and generalized Hodge–de Rham decompositions *Demonstratio Math.* **18** 77–89
[64] Rubenthaler H 1994 Les Paires Duales Dans les Algèbres de Lie Réductives *(Astérisque Soc. Math. France)* p 219
[65] Rudolph G, Schmidt M and Volobuev I P 2000 Classification of gauge orbit types for SUn-gauge theories *Preprint* math-ph/0003044 *(J. Math. Phys. Anal. Geom.* at press)
[66] Rudolph G, Schmidt M and Volobuev I P 2002 Partial ordering of gauge orbit types for SUn-gauge theories *J. Geom. Phys.* **42** 106–38
[67] Rudolph G and Schmidt M 2001 On a certain stratification of the gauge orbit space *Preprint* math-ph/0104026 (*Rep. Math. Phys.* at press)
[68] Schmidt M 1999 Classification and partial ordering of reductive Howe dual pairs of classical Lie groups *J. Geom. Phys.* **29** 283–318
[69] Shabanov S V 1995 2D Yang Mills theories, gauge orbit spaces and the path integral quantization *Commun. Theor. Phys. (Allahabad)* **4** 1–62

[70] Singer I M 1978 Some remarks on the Gribov ambiguity *Commun. Math. Phys.* **60** 7–12
[71] Singer I M 1981 The geometry of the orbit space for non-Abelian gauge theories *Phys. Scr.* **24** 817–20
[72] Sjamaar R and Lerman E 1991 Stratified symplectic spaces and reduction *Ann. Math.* **134** 375–422
[73] Steenrod N 1951 *The Topology of Fibre Bundles* (Princeton, NJ: Princeton University Press)
[74] t'Hooft G 1978 On the phase transition towards permanent quark confinement *Nucl. Phys.* B **138** 1–25
[75] Trautman A 1970 Fibre bundles associated with space-time *Rep. Math. Phys.* **1** 29–62
[76] van Baal P 1992 More (thoughts on) Gribov copies *Nucl. Phys.* B **369** 259–75
[77] van Baal P and van den Heuvel B 1994 Zooming in on the SU(2) fundamental domain *Nucl. Phys.* B **417** 215–37
[78] Witten E 1982 An SU(2)-anomaly *Phys. Lett.* B **117** 324–8
[79] Zwanziger D 1994 Fundamental modular region, Boltzmann factor, and area law in lattice theory *Nucl. Phys.* B **412** 657–730